Perturbation problems in homogenization of hamilton-jacobi equations

Pierre Cardaliaguet, Claude Le Bris, Panagiotis Souganidis

PERTURBATION PROBLEMS IN HOMOGENIZATION OF HAMILTON-JACOBI EQUATIONS

PIERRE CARDALIAGUET, CLAUDE LE BRIS AND PANAGIOTIS E. SOUGANIDIS

Version: January 15, 2017

Abstract. This paper is concerned with the behavior of the ergodic constant associated with convex and superlinear Hamilton-Jacobi equation in a periodic environment which is perturbed either by medium with increasing period or by a random Bernoulli perturbation with small parameter. We find a first order Taylor's expansion for the ergodic constant which depends on the dimension d. When $d = 1$ the first order term is non trivial, while for all $d \geq 2$ it is always 0. Although such questions have been looked at in the context of linear uniformly elliptic homogenization, our results are the first of this kind in nonlinear settings. Our arguments, which rely on viscosity solutions and the weak KAM theory, also raise several new and challenging questions.

1. Introduction

The paper is concerned with the behavior of the ergodic constant associated with convex and superlinear Hamilton-Jacobi (HJ for short) equations in a periodic environment which is perturbed either by medium with increasing period which is a multiple of the original one or by a random Bernoulli perturbation with small parameter. We find a first-order Taylor's expansion for the ergodic constant which depends on the dimension d. When $d = 1$ the first order term is non trivial, while for all $d \geq 2$ it is always 0. Our results are the first of this kind for nonlinear problems. The arguments, which rely on viscosity solutions and the weak KAM theory, also raise several new and challenging questions.

The motivation for this work came from the recent studies by Anantharaman and Le Bris [2, 3] and Duerinckx and Gloria [11], who considered similar questions for linear uniformly elliptic operators (and systems in [11]). The former paper considered Bernoulli perturbations of a periodic environment, while the latter reference, which complemented and generalized the work of the former, considered Bernoulli perturbations of a stationary ergodic medium and provided, taking strong advantage of the linearity of the equation, a full expansion.

Loosely speaking the aim of homogenization is to replace a possibly complicated heterogeneous medium with a homogeneous environment that shares the same macroscopic properties. In concrete models (equations) it allows to eliminate the fine scale up to an error which is controlled by the size of fine scale as compared to the macroscopic size.

From the modeling point of view, assuming that the medium is periodic is a rather rigid and idealistic assumption and somewhat remote from actual settings. Indeed, in view of the industrial process they are produced by, manufactured media, such as composite materials, can be considered, under reasonable conditions, to be periodic or at least "approximately"

Cardaliaguet was partially supported by the ANR (Agence Nationale de la Recherche) project ANR-12-BS01-0008-01. Souganidis was partially supported by the National Science Foundation Grants DMS-1266383 and DMS-1600129. Part of the work was completed during Le Bris visits to the University of Chicago.

periodic. However, natural media, such as the subsoil, have no reason whatsoever to be periodic. Periodicity is then a mathematical idealization, or artifact, that might lead to inaccurate results.

A well established option is then to consider the medium to be random, and, more precisely, stationary ergodic. This assumption conveniently makes up for the absence of periodicity, and, actually, includes periodicity as a particular case. The mathematical theory of random homogenization, both quantitative and qualitative, born in the early 1970s, has seen an enormous growth over the past fifteen years. However in spite of the appeal the theory, its application to actual media for real applications and, in particular, numerical simulations, remains a challenging issue. Random homogenization, and all approaches that derive from it, may indeed be computationally prohibitively expensive, even for the simplest possible equations arising, for instance, in the engineering sciences. A compromise between the economical but idealistic periodic and the more general but extremely costly random settings is to consider small random perturbations of periodic scenarios. The response of the medium in terms of this small perturbation, that is the modification of the homogenized limit in the presence of the small random perturbation, is intuitively expected to be easier to evaluate. This was shown to be indeed true in the case of homogenization of linear elliptic equations in [2, 3]. A formal derivation of the first-order perturbation and numerical experiments performed there confirmed that it is possible, at a much reduced computational price, to approximate the homogenized limit of the random problem. As mentioned above, the approach has then been proven to be rigorous, and extended to all orders of perturbation, in a subsequent publication [11].

In order to convey to the reader the flavor of the mathematical mechanism in action, we consider the following simplistic setting, which can be thought as a computational model for the whole space $\mathbb{R}^d$. Let $F_{per}(x) := \sum_{k\in\mathbb{Z}^d} v(\cdot - k)$ be a $\mathbb{Z}^d$-periodic function that repeats itself within a presumably extremely large box of size R, and assume that a certain output, $\mathcal{S}_{per}$, is computed from it. In the specific case addressed in [2, 3], F_{per} and $\mathcal{S}_{per}$ were respectively the matrix valued coefficient A_{per} of the linear elliptic operator $-\mathrm{div}(A_{per}(./\varepsilon)\,\nabla)$ and the matrix $\overline{A}$ replacing A_{per} in the homogenized limit. In this paper, the function F_{per} is the periodic Hamiltonian of a Hamilton-Jacobi equation and the outcome $\mathcal{S}_{per}$ is the homogenized Hamiltonian $\overline{H}$. Assume now that F_{per} is perturbed by the addition of a random function of, for example, the form $\zeta_\eta(x) := \sum_{k\in\mathbb{Z}^d} X_k\zeta(\cdot - k)$, where the X_k's are Bernoulli random variables of a small parameter η, which are all independent from one another. Intuitively, at first order in η, the perturbation experienced by F_{per} consists of adding exactly one ζ at each possible location within the large box of size R. The probability of having two distinct non zero variables X_k is of order η^2, a term negligible with respect to the first order term in η. The perturbation of the outcome $\mathcal{S}$ with respect to the outcome $\mathcal{S}_{per}$ can therefore be calculated using only the configurations of the periodic medium perturbed in one random location. Of course, the above argument is formal in many respects. For the rigorous result, we must consider the whole infinite space $\mathbb{R}^d$ instead of a large box of finite size and need to prove the fact that all other configurations than those with exactly one non zero X_k do not contribute to the asymptotics at first order. But the underlying idea remains. This general discussion is made more precise below.

We describe next in a somewhat informal way the results of the paper. The actual statement need hypotheses which will be given in Section 2.

Let $H := H(p,x)$ be a Hamiltonian which is coercive in p and $\mathbb{Z}^d-$periodic in x. It was shown by Lions, Papanicolaou and Varadhan [19] that there exists a unique $\overline{H}$, often referred to as the effective Hamiltonian or the ergodic constant, such that the cell problem

$$H(D\chi, x) = \overline{H} \text{ in } \mathbb{R}^d, \tag{1.1}$$

has a continuous, $\mathbb{Z}^d-$periodic (viscosity) solution χ known as a corrector.

Correctors are obviously not unique. Throughout the paper, we make the normalization that $\chi(0) = 0$.

We recall that $\overline{H}$ is obtained as the uniform limit, as $\delta \to 0$, of $-\delta v^\delta$, where v^δ is the unique periodic solution to the approximate cell problem

$$\delta v^\delta + H(Dv^\delta, x) = 0 \text{ in } \mathbb{R}^d. \tag{1.2}$$

We consider two types of perturbations. The first is also periodic with increasingly large integer period. The second is random (Bernoulli) with small intensity.

In the first case the perturbed $R\mathbb{Z}^d$-periodic Hamiltonian H_R, with $R \in \mathbb{N}$, is

$$H_R(p,x) := H(p,x) - \zeta_R(x), \tag{1.3}$$

with the R-periodic function $\zeta_R : \mathbb{R}^d \to \mathbb{R}$ defined as

$$\zeta_R(x) := \sum_{k\in\mathbb{Z}^d} \zeta(x - Rk), \tag{1.4}$$

where

$$\zeta : \mathbb{R}^d \to \mathbb{R} \;\text{ is nonnegative, Lipschitz continuous and compactly supported.} \tag{1.5}$$

In view of the form of ζ_R, we often refer to $\zeta(\cdot - Rk)$ as a "bump" located at the point Rk.

Let $\overline{H}_R$ be the ergodic constant associated with H_R. Then there exists a continuous $R\mathbb{Z}^d-$periodic solution χ_R of the cell-problem

$$H(D\chi_R, x) = \zeta_R(x) + \overline{H}_R \text{ in } \mathbb{R}^d, \tag{1.6}$$

which is "renormalized" by $\chi_R(0) = 0$.

Since, as $R \to +\infty$, there are fewer bumps in a given ball, it is reasonable to expect that, as $R \to +\infty$, $\overline{H}_R$ converges to $\overline{H}$. Our goal is to obtain quantitative information (rate, first term in the expansion) for this convergence.

In the second type of perturbation, the randomly perturbed Hamiltonian H_η is given by

$$H_\eta(p,x) := H(p,x) - \zeta_\eta(x) \tag{1.7}$$

where

$$\zeta_\eta(x) := \sum_{k\in\mathbb{Z}^d} \zeta(x-k) X_k, \tag{1.8}$$

with ζ satisfying (1.5) and

$$(X_k)_{k\in\mathbb{Z}^d} \text{ a family of i.i.d. Bernoulli random variables of parameter } \eta. \tag{1.9}$$

Contrary to the periodic setting, in random media the effective Hamiltonian is not characterized by the cell-problem. The reason is that to guarantee its uniqueness, it is necessary to have correctors which are strictly sub-linear at infinity. As shown in Lions and Souganidis [23], in general, this is not possible.

The effective constant $\overline{H}_\eta$ is defined, for instance, through the discounted problem

$$\delta v^{\eta,\delta} + H_\eta(Dv^{\eta,\delta}, x) = 0 \text{ in } \mathbb{R}^d$$

which has unique bounded solution $v^{\eta,\delta}$, as the almost sure limit (see Souganidis [28])

$$\overline{H}_\eta := \lim_{\delta\to 0} -\delta v^{\eta,\delta}(0).$$

Note that, as $\eta \to 0$, the probability that there is a bump in a fixed ball becomes smaller and smaller. So here again it is natural to expect that $\overline{H}_\eta$ converges to $\overline{H}$ as $\eta \to 0$ and we want to understand at which rate this convergence holds.

We establish two types of results. The first is an estimate of the difference between $\overline{H}_R$ or $\overline{H}_\eta$ and $\overline{H}$, which holds even for more general (almost periodic) perturbations.

We prove that, if $H = H(p, x)$ is convex and coercive in p and $\mathbb{Z}^d-$periodic in x, then there exists $C > 0$ depending only on ζ (see Corollary 3.2 and Corollary 3.4) such that

$$0 \le \overline{H} - \overline{H}_R \le CR^{-d} \text{ for all } R \in \mathbb{N}, \tag{1.10}$$

$$0 \le \overline{H} - \overline{H}_\eta \le C\eta \text{ for all } \eta \in (0, 1), \tag{1.11}$$

and, in particular,

$$\lim_{R\to\infty} \overline{H}_R = \overline{H} \;\text{ and }\; \lim_{\eta\to 0} \overline{H}_\eta = \overline{H}. \tag{1.12}$$

The result is unusual in the homogenization of Hamilton-Jacobi equations because the perturbations do not vanish in the L^∞-norm and relies strongly on the fact that the bumps are nonnegative. In general the convergence does not hold otherwise; see Achdou and Tchou [1], Lions [18] and Lions and Souganidis [24], where we also refer for more general statements about homogenization with fixed perturbations of periodic and random environments.

We point out that (1.10), (1.11) and (1.12) are examples of more general statements which hold for general almost periodic or random perturbations; see Propositions 3.1 and 3.3.

In view of (1.10), (1.11) and (1.12), it is natural, and this is the second type of results in this paper, to identify the limits

$$\lim_{R\to\infty} R^d(\overline{H}_R - \overline{H}) \text{ and } \lim_{\eta\to 0} \eta^{-1}(\overline{H}_\eta - \overline{H}).$$

It turns out that is much more complicated than proving (1.12) and we only have a complete answer under some additional assumptions.

In order to describe the results as well as to give a hint of the subtlety, we explain briefly and very informally the proof of (1.10). Similar arguments justify (1.11).

We argue as if both χ and χ_R were smooth, which is not the case in general. We subtract (1.1) from (1.6), we linearize the difference around $D\chi$ assuming also that H is smooth, and we use the convexity of H to find

$$\overline{H}_R - \overline{H} \ge D_pH(D\chi, x) \cdot D(\chi_R - \chi) + \zeta_R, \tag{1.13}$$

where D_pH denotes the gradient of H with respect to p.

Let $\widetilde{\sigma}$ be the invariant measure associated with (1.1), which exists in view of the weak KAM theory (see Fathi [14]), that is, $\widetilde{\sigma}$ is a Borel probability measure in the unit cube $[-1/2, 1/2)^d$ and

$$-\mathrm{div}\,(\widetilde{\sigma} D_pH(D\chi, x)) = 0.$$

We extend $\widetilde{\sigma}$ by periodicity to $\mathbb{R}^d$ and we integrate both sides of (1.13) with respect to $\widetilde{\sigma}$ over $[-R/2, R/2)^d$. Using the fact that, for R large enough, there is only the compactly supported bump ζ in the cube $[-R/2, R/2)^d$, we find

$$R^d(\overline{H}_R - \overline{H}) \geq -\int_{\mathbb{R}^d} \zeta(x) d\widetilde{\sigma}(x). \tag{1.14}$$

The last inequality not only justifies the right-hand side of (1.10), but also hints that the limit of $R^d(\overline{H}_R - \overline{H})$ should be $-\int_{\mathbb{R}^d} \zeta d\widetilde{\sigma}$. This turns out to be false.

Indeed, under some assumptions on the minimizing Mather measure in the weak KAM formulation of (1.1) which are stated informally below, we show in Theorem 4.1, that, when $d = 1$,

$$\lim_{R\to+\infty} R(\overline{H}_R - \overline{H}) = -\left(\int_{-1/2}^{1/2} \frac{1}{D_pH(\chi'(x), x)} dx\right)^{-1} \int_{\mathrm{sppt}(\zeta)} \left(H^{-1}(\zeta(x) + \overline{H}, x) - H^{-1}(\overline{H}, x)\right) dx,$$

and, when $d \geq 2$,

$$\lim_{R\to+\infty} R^d(\overline{H}_R - \overline{H}) = 0.$$

For the proof we assume that the invariant measure is unique, has a non vanishing rotational number and its marginal $\widetilde{\sigma}$ has a full support. The assumption on $\widetilde{\sigma}$ is strong and holds only for specific classes of Hamiltonian.

A schematic view of our strategy of proof goes as follows. Let L be the convex dual of H defined in (2.3). The variational interpretation of (1.6) and (1.1) implies that the respective correctors χ_R and χ satisfy the identities

$$\chi_R(x) = \inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot{\gamma}(s), \gamma(s)) + \overline{H}_R + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t))\right],$$

and

$$\chi(x) = \inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot{\gamma}(s), \gamma(s)) + \overline{H})\, ds + \chi(\gamma(t))\right],$$

where $\mathcal{A}_x$ is the set of Lipschitz curves in $\gamma : [0, \infty) \to \mathbb{R}^d$ such that $\gamma(0) = x$.

Let $\overline{\gamma}^x$ denote the optimal path in the expression for χ, which exists in view of the assumptions on H. Then based on the equalities above, the difference of the two Hamiltonians $\overline{H}_R$ and $\overline{H}$ reads, for all $t > 0$, as

$$\begin{aligned} t(\overline{H}_R - \overline{H}) \;&=\; -\inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot{\gamma}(s), \gamma(s)) + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t))\right] + \chi_R(x) \\ &\quad + \int_0^t L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s))\, ds + \chi(\overline{\gamma}^x(t)) - \chi(x). \end{aligned}$$

Identifying the limit of $R^d(\overline{H}_R - \overline{H})$ therefore amounts to constructing a specific trajectory that almost minimizes the infimum problem in the right-hand side. Clearly, that infimum is not achieved by $\overline{\gamma}^x$, since the presence of ζ_R has perturbed the original problem. However, $\overline{\gamma}^x$ is expected to provide an accurate approximation of the infimum, at least far from the bumps. The actual proof consists in making this intuition precise and in understanding the behavior of the optimal trajectories near the bumps.

The same strategy of proof applies to the random perturbation. There it is necessary to construct an appropriate random perturbation of the trajectory $\overline{\gamma}^x$. Most of the argument then aims at fixing all the necessary technicalities in the construction of that particular modified trajectory.

An intuitive way to explain the result is that, when $d \geq 2$, the minimizers in (1.6) eventually avoid the bump and stay close to those of (1.1), while, when $d = 1$, they must pass through the bump. A similar interpretation can be used for the result in the random setting.

The conclusion that, when $d \geq 2$, $\overline{H}_R$ does not deviate much from $\overline{H}$ is in stark contrast with what is happening for uniformly elliptic divergence form operators where the first term in the expansion is nonzero. The heuristic explanation for this difference is that in the Hamilton-Jacobi setting information is propagated along curves which are lower dimensional objects when $d \geq 2$, while for the elliptic problem the information is obtained by averaging.

Next we describe some of the major ingredients in our analysis concentrating always for simplicity on the periodic problem. An important fact is that, after a renormalization by additive constants, the correctors χ_R of the periodically perturbed cell-problem (1.6) converge, along subsequences as $R \to +\infty$ and locally uniformly in $\mathbb{R}^d$, to solutions χ_∞, which are no longer periodic, of the equation

$$H(D\chi_\infty, x) = \zeta(x) + \overline{H} \text{ in } \mathbb{R}^d; \tag{1.15}$$

the existence of such solutions was also proved by different methods in [1], [18] and [24].

The interesting property of χ_∞ is that it keeps track of the perturbed problem, in the sense that, at least formally (see Lemma 4.2 for a rigorous statement),

$$0 \geq \liminf_{R\to+\infty} R^d(\overline{H}_R - \overline{H}) \geq - \int_{\mathbb{R}^d} \langle D_p H(D\chi, x), D\chi_\infty - D\chi\rangle d\widetilde{\sigma}(x). \tag{1.16}$$

It follows from the invariance property of $\widetilde{\sigma}$ that the right-hand side of (1.16) sees only the difference of $\chi_\infty - \chi$ at infinity.

The analysis of χ_∞ is in itself a very intriguing problem. Using that $d \geq 2$ we prove in Corollary 2.4 that there exists a constant c such that χ_∞ is always above $\chi + c$, coincides with $\chi + c$ outside of a "cylinder", and tends to $\chi + c$ at infinity. This is enough to show that the right-hand side of (1.16) vanishes, which in turn proves that $R^d(\overline{H}_R - \overline{H})$ tends to 0. The analysis when $d = 1$ is based on a more direct argument. The proof for the random perturbed problem relies on structure of χ_∞ as well.

We continue with a rather brief summary of the history of the problem acknowledging that is really not possible to refer to all previous papers. As already mentioned earlier the first homogenization result for Hamilton-Jacobi equations in periodic environments was proved in [19]. Subsequent developments are due to Evans [12, 13] and, among others, Majda and Souganidis [25]. The first result about the homogenization of Hamilton-Jacobi equations in random media was obtained in Souganidis [28] and Rezakhanlou and Tarver [26]. Other important contributions to the subject always in the context of the qualitative theory of homogenization for Hamilton-Jacobi equations are Lions and Souganidis [21, 22, 23], Armstrong and Souganidis [6, 7], and Cardaliaguet and Souganidis [8, 9]. Quantitative results, that is error estimates, were shown in Armstrong, Cardaliaguet and Souganidis [5] and Armstrong and Cardaliaguet [4].

Organization of the paper. The paper is organized as follows. In the next section we introduce the main assumptions and recall some well known facts from the weak KAM theory. In Section 3 we state and prove two general results about the growth of the perturbations of the ergodic constant and make the connections with (1.10) and (1.11). In Section 4 we introduce the assumptions and state and prove the asymptotic result for periodic perturbations, while in Section 5 we consider random perturbations.

Notation and terminology. We work in $\mathbb{R}^d$ and we write $|x|$ for the Euclidean length of a vector $x \in \mathbb{R}^d$ and, for $x, y \in \mathbb{R}^d$, $\langle x, y\rangle$ is the usual inner product. The sets of integers and positive and nonnegative integers are respectively $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$. If $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, then $|k|_\infty := \max_{1\le i\le d} |k_i|$. The cube centered at $x \in \mathbb{R}^d$ and of size $R > 0$ is denoted by $Q_R(x) := x + [-R/2, R/2)^d$ and we set $Q_R := Q_R(0)$ and $Q := Q_1$ for simplicity. Given a finite subset $A \subset \mathbb{Z}^d$, $\sharp A$ denotes the number of elements of A. Given a nonnegative measure μ and function ζ, $\mathrm{sppt}(\mu)$ and $\mathrm{sppt}(\zeta)$ are respectively their support. If $f : \mathbb{R}^d \to \mathbb{R}$ is bounded, $\mathrm{osc} f := \sup_{\mathbb{R}^d} f - \inf_{\mathbb{R}^d}$. If f is integrable and $E \subset \mathbb{R}^d$ has a finite volume, we denote by $⨍_E f$ the average of f on E, that is $⨍_E f = |E|^{-1} \int_E f$. For notational convenience, we write $A \lesssim B$, if $A \le CB$ for some $C > 0$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$. Given $f : [a, b] \to \mathbb{R}$, $[f]_a^b := f(b) - f(a)$, and, for all $k \in \mathbb{N} \cup \{\infty\}$, $C_c^k(\mathbb{R}^m)$ is the set of compactly supported C^k real valued functions on $\mathbb{R}^m$. Throughout the paper, C is a constant that may vary from line to line and depends on the Hamiltonian H and the space dimension d, unless otherwise specified. All the Hamilton-Jacobi equations encountered in the text have to be understood in the sense of viscosity solutions [10].

The random setting. We describe here the random setting that we use in the paper and introduce the necessary notation and terminology.

The general setting is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we write $\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ for the expectation of a random variable $X \in L^1(\Omega; \mathbb{R})$. We assume that the group $(\mathbb{Z}^d, +)$ acts on Ω. We denote by $(\tau_k)_{k\in\mathbb{Z}^d}$ this action and assume that it is measure preserving, that is, for all $k \in \mathbb{Z}^d$ and $A \in \mathcal{F}$, $\mathbb{P}[\tau_k A] = \mathbb{P}[A]$, and ergodic, that is, for any translation invariant $A \in \mathcal{F}$, $\mathbb{P}[A] = 0$ or 1.

A process $F : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be $\mathbb{Z}^d$-stationary if, for all $k \in \mathbb{Z}^d$, $F(x + k, \omega) = F(x, \tau_k\omega)$ almost everywhere in x and almost surely in ω.

The ergodic theorem says that, if $F \in L^\infty(\mathbb{R}^d; L^1(\Omega))$ is stationary, then, as $N \to \infty$,

$$\frac{1}{(2N+1)^d} \sum_{|k|_\infty \le N} F(x, \tau_k\omega) \to \mathbb{E}(F(x, \cdot)) \text{ for any } x \in \mathbb{R}^d \text{ and almost surely in } \omega.$$

Finally we remark, although we will not be making use of this in the paper, that almost periodic functions can be thought as stationary functions in an appropriate probability space with continuous stationary and ergodic action (translation).

2. The assumptions and some basic facts

We introduce the assumptions on the Hamiltonian H and recall some basic facts from the weak KAM theory, for which we refer to [14]. We discuss the one-dimensional setting in detail as well as the existence and properties of the "corrector" of the perturbed problem in the whole space.

The motivation for this presentation is to have all assumptions and their immediate consequences in one place. We recommend, however, that the reader skips this section the first time and returns to it as is necessary while going through the other parts of the paper.

Assumptions on the Hamiltonian. We assume that $H \in C^2(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ is

$$\begin{cases} \mathbb{Z}^d\text{−periodic in the second variable, that is} \\ H(p, x+k) = H(p,x) \text{ for all } p \in \mathbb{R}^d, x \in \mathbb{R}^d \text{ and } \ k \in \mathbb{Z}^d, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \text{strictly convex and super-linear with respect to the first one, that is} \\ \qquad D^2_{pp}H > 0 \text{ and} \lim_{|p|\to+\infty} |p|^{-1}H(p,x) = +\infty \text{ uniformly in } x. \end{cases} \tag{2.2}$$

Facts from the weak KAM theory. Let L be the Lagrangian associated with H which, for all $(\alpha, x) \in \mathbb{R}^d \times \mathbb{R}^d$, is given by

$$L(\alpha, x) := \sup_{p\in\mathbb{R}^d} \left\{ -\langle p, \alpha\rangle - H(p,x) \right\}. \tag{2.3}$$

We recall from the introduction that $\overline{H}$ is the effective (ergodic) constant associated with H, that is $\overline{H}$ is the unique constant such that the cell problem (1.1) has a $\mathbb{Z}^d$−periodic, continuous solution χ. Note that the coercivity of H yields that χ is Lipschitz continuous. The weak KAM theory provides an alternative characterization for $\overline{H}$, namely

$$-\overline{H} = \inf_{\mu} \int_{\mathbb{R}^d\times Q} L(\alpha, x) d\mu(\alpha, x), \tag{2.4}$$

where the infimum is taken over the Radon measures μ on $\mathbb{R}^d \times \mathbb{R}^d$ which are $\mathbb{Z}^d$−periodic in x, have weight 1 on $\mathbb{R}^d \times Q$ and are closed, that is, for all $\mathbb{Z}^d$-periodic $\phi \in C^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d\times Q} \langle D\phi(x), \alpha\rangle d\mu(\alpha, x) = 0.$$

Throughout the text, we often write $\widetilde{\mu}$ and $\widetilde{\sigma}$ for an optimal measure in the minimization problem (2.4) and its marginal with respect to x respectively. In the context of the weak KAM theory such $\widetilde{\mu}$ and $\widetilde{\sigma}$ are called respectively a minimizing Mather measure and a projected minimizing measure. Note that the restriction of $\widetilde{\sigma}$ to Q is a probability measure.

We use the following well known facts from the weak KAM theory (see [14]):

$$\text{any corrector } \chi \text{ is } \widetilde{\sigma}\text{−a.e. differentiable},$$

$$\widetilde{\mu} \text{ is the image of the measure } \widetilde{\sigma} \text{ by the map } x \to (D_pH(D\chi(x), x), x),$$

and

$$\widetilde{\sigma} \text{ is an invariant measure for the flow generated by the vector field } x \mapsto -D_pH(D\chi(x), x),$$

that is

$$\operatorname{div}\left(\widetilde{\sigma} D_pH(D\chi, x)\right) = 0 \text{ in the sense of distributions in } \ \mathbb{R}^d. \tag{2.5}$$

An assumption on the Mather measure and its consequences. In order to prove the asymptotic results and, in particular, the existence of the limits discussed in the Introduction, we need to further assume that

$$\begin{cases} \text{the projected Mather measure } \widetilde{\sigma} \text{ associated with } H \text{ is unique,} \\ \text{has a nonzero rotation number } e \text{ and full support in } \mathbb{R}^d. \end{cases} \tag{2.6}$$

The assumption of the full support in $\mathbb{R}^d$ is written as

$$\operatorname{sppt}(\widetilde{\sigma}) = \mathbb{R}^d,$$

and the nonzero rotation number e is given by

$$e := \int_Q -D_pH(D\chi(x), x)d\widetilde{\sigma}(x) \neq 0. \tag{2.7}$$

The first two conditions in (2.6), that is the uniqueness of $\widetilde{\sigma}$ and existence of a nonzero rotation number, are rather mild. For example, if $H(p, x) = \widetilde{H}(p + \overline{p}, x)$ for some Hamiltonian $\widetilde{H}$ satisfying (2.1) and (2.2) and some $\overline{p} \in \mathbb{R}^d$, then $\widetilde{\sigma}$ is unique for a "generic" $\overline{p}$ and the nonzero rotation number exists for $\overline{p}$ large enough; see [14]. That the projected Mather measure has full support in $\mathbb{R}^d$ is a much stronger assumption and only holds under restrictive structure conditions.

We continue listing several consequences of (2.6) that are used in the rest of the paper. We refer to [14] and references therein for the proofs.

Since the projected Mather measure has a full support,

$$\text{the projected Aubry set is } \mathbb{R}^d. \tag{2.8}$$

It then follows that

$$\text{any corrector } \chi \text{ is of class } C^1, \text{ and, thus, also } C^{1,1}. \tag{2.9}$$

The strict convexity of the Hamiltonian also implies that

$$\text{the correctors are unique up to an additive constant.} \tag{2.10}$$

Indeed, if χ and $\widetilde{\chi}$ are two correctors, subtracting their respective equations and using the strict convexity we find, for some $C > 0$,

$$0 = H(D\widetilde{\chi}, x) - H(D\chi, x) \geq \langle D_pH(D\chi, x), D(\widetilde{\chi} - \chi)\rangle + C|D(\widetilde{\chi} - \chi)|^2.$$

Multiplying the inequality above by $\widetilde{\sigma}$, integrating over Q with respect to $\widetilde{\sigma}$ and integrating by parts using the periodicity and the fact that $\widetilde{\sigma}$ is an invariant measure, that is (2.5) holds, we obtain

$$0 \geq \int_Q |D(\widetilde{\chi} - \chi)|^2 d\widetilde{\sigma}(x).$$

Thus the continuous maps $D\widetilde{\chi}$ and $D\chi$ agree on a dense subset of $\mathbb{R}^d$ and therefore everywhere.

In view of (2.9), we can define the flow $\overline{\gamma}^x : \mathbb{R} \to \mathbb{R}^d$ of optimal trajectories for any initial position $x \in \mathbb{R}^d$ by

$$\dot{\overline{\gamma}}^x(t) = -D_pH(D\chi(\overline{\gamma}^x(t)), \overline{\gamma}^x(t)) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \; \overline{\gamma}^x(0) = x;$$

we note that the map $x \mapsto \overline{\gamma}^x(t)$ is continuous for any t.

We recall now that the optimality of $\overline{\gamma}^x$ implies that, for all $t \geq 0$,

$$\chi(x) = \int_0^t (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \overline{H})ds + \chi(\overline{\gamma}^x(t)). \tag{2.11}$$

The uniqueness of the projected Mather measure implies that it is actually ergodic. As a result, for $\widetilde{\sigma}-$ a.e. $x \in \mathbb{R}^d$, we have

$$\lim_{t\to\pm\infty} \frac{\overline{\gamma}^x(t)}{t} = \int_{\mathbb{T}^d} -D_pH(D\chi(x), x)d\widetilde{\sigma}(x) = e. \tag{2.12}$$

As a matter of fact we will see below (Lemma 2.2), that, as a consequence of the uniqueness of $\widetilde{\sigma}$, (2.12) actually holds for all $x \in \mathbb{R}^d$.

We present now a simple example satisfying (2.1), (2.2) and (2.6). Let $H(p,x) = |p+\overline{p}|^2$ for some non rational direction $\overline{p} \in \mathbb{R}^d$. In this case, we have $\chi = 0$ and $\overline{\gamma}^x(t) = x + t\overline{p}$. The unique invariant measure is $\widetilde{\sigma} = 1$ and $e = -2\overline{p}$.

The KAM theory then implies that (2.6) holds true for $H(p,x) := |p+\overline{p}|^2 - V(x)$ with $\overline{p}$ a Diophantine vector and $V : \mathbb{R}^d \to \mathbb{R}$ periodic, smooth and small enough. Obviously, (2.1) and (2.2) are satisfied.

The one-dimensional setting when (2.1)**,** (2.2) **and** (2.6) **hold.** We know from (2.6) that the cell problem (1.1) has a $\mathbb{Z}$-periodic solution $\chi \in C^{1,1}(\mathbb{R})$ and $\int_Q \chi'(x)dx = 0$.

The strict convexity of H implies that the inverse $H^{-1}(\cdot,x)$ of $H(\cdot,x)$ has two branches $H^{-1}_{\pm}(\cdot,x)$ as long as one is away from its minimum, which is the case in view of (2.6). Since the corrector is smooth, $\chi'(x)$ must be, for all x, in the same branch of $H^{-1}(\cdot,x)$ and we can rewrite (1.1) as an the ode

$$\chi'(x) = H^{-1}(\overline{H}, x), \tag{2.13}$$

using only one of them. The choice of the branch, which from now we denote as $H^{-1}(\cdot,x)$, is dictated by $\int_Q \chi'(x)dx = 0$.

In view of the above discussion, in any Q_R with $R \in \mathbb{Z}$, we have

$$\int_{Q_R} H^{-1}(\overline{H}, x)dx = 0. \tag{2.14}$$

It also follows from (2.6) and (2.5) that the invariant measure $\widetilde{\sigma}$ associated with the cell problem at hand has $\mathbb{Z}-$periodic extension in $\mathbb{R}$ with density

$$\widetilde{\sigma}(x) = (\int_Q \frac{1}{D_pH(\chi'(y),y)}dy)^{-1}\frac{1}{D_pH(\chi'(x),x)}; \tag{2.15}$$

note that for notational simplicity we often identify the invariant measure with its density,

Let $D_rH^{-1}(\cdot,x)$ denote the derivative of $r \mapsto H^{-1}(r,x)$ with respect to the first argument. It follows from (2.13) that

$$D_rH^{-1}(\overline{H},x) = \frac{1}{D_pH(\chi'(x),x)}, \tag{2.16}$$

and, in view of (2.15),

$$D_rH^{-1}(\overline{H},x) = \left(\int_Q \frac{1}{D_pH(\chi'(y),y)}dy\right)\ \widetilde{\sigma}(x). \tag{2.17}$$

We conclude with the following classical example always for $d = 1$. The Hamiltonian is $H(p,x) = |p+\overline{p}|^2 - V(x)$ for some fixed $\overline{p} \in \mathbb{R}$ and a $\mathbb{Z}$-periodic potential V with $\min_{x\in Q} V(x) = 0$. It is well known that, if $|\overline{p}| \geq \int_Q \sqrt{V(x)}dx$, then the cell problem

$$|\chi_x + \overline{p}|^2 = V(x) + \overline{H} \text{ in } \mathbb{R},$$

has a smooth $\mathbb{Z}-$periodic solution for $\overline{H}$ given by $|\overline{p}| = \int_Q \sqrt{V(x)+\overline{H}}$. This last expression and the sign of $\overline{p}$ identify the branch of the $\sqrt{\cdot}$ that we need to choose.

A corrector χ_∞ of the perturbed problem in the whole space. An important ingredient in our analysis is the construction of a "perturbed corrector" χ_∞, that is a solution to (1.15), which, as it turns out (see Lemma 4.2), keeps track of the difference between $\overline{H}_R$ and $\overline{H}$ as $R \to \infty$.

The first step in finding χ_∞ is to obtain independent of R sup- and Lipschitz bounds for the $R\mathbb{Z}^d-$ periodic solutions χ_R to (1.6); recall that we always consider $R \in \mathbb{N}$.

Lemma 2.1. *Assume* (2.1), (2.2) *and* (1.5). *There exist solutions χ_R of the perturbed cell problem* (1.6) *such that*

$$\|\chi_R\|_\infty + \|D\chi_R\|_\infty \lesssim 1.$$

Proof. The gradient bound follows immediately from the coercivity of H and holds for any solution of the cell problem. For the L^∞-bound, we consider the approximate cell problems (1.2) and

$$\delta v_R^\delta + H(Dv_R^\delta, x) = \zeta_R \text{ in } \mathbb{R}^d, \tag{2.18}$$

which are respectively $\mathbb{Z}^d$ and $R\mathbb{Z}^d$ periodic. Since, for any $\varepsilon > 0$, $v_R^\delta - \varepsilon$ is a strict subsolution to (1.2) in $\mathbb{R}^d \backslash \text{sppt}(\zeta_R)$, the maximum of $v_R^\delta - \varepsilon - v^\delta$, if positive, can only be reached at some $x_\varepsilon \in \text{sppt}(\zeta_R)$ and, in view of the periodicity, we may assume that $x_\varepsilon \in \text{sppt}(\zeta)$. Similarly, the minimum of $v_R^\delta - \varepsilon - v^\delta$, if negative, is reached at a point $y_\varepsilon \in \text{sppt}(\zeta)$.

Thus

$$\text{osc}_{\mathbb{R}^d}(v_R^\delta - v^\delta) \leq \text{osc}_{\text{sppt}(\zeta)}(v_R^\delta - v^\delta) + 2\varepsilon \leq \|D(v_R^\delta - v^\delta)\|_\infty \text{diam}(\text{sppt}(\zeta)) + \varepsilon. \tag{2.19}$$

Recall that the v^δ's are $\mathbb{Z}^d-$periodic. Moreover, in view of the assumed coercivity and bounds on H, the v^δ's are Lipschitz continuous uniformly in δ. Hence their oscillations are bounded uniformly in δ.

It follows from (2.19) that the oscillation of v_R^δ is also bounded, uniformly with respect to R and δ. Thus, we can extract a subsequence $\delta_n \to 0$ such that $v_R^{\delta_n} - v_R^{\delta_n}(0)$ converge uniformly in $\mathbb{R}^d$ to a solution χ_R of the perturbed cell problem (1.6) satisfying the uniform L^∞ and Lipschitz bounds.

Up to a subsequence, we can assume that, as $R \to \infty$, the χ_R's converge locally uniformly to some χ_∞, which is no longer periodic, solving (1.15). ☐

We discuss next some properties of the map χ_∞ and the optimal trajectories for χ and χ_∞ which will be useful for the asymptotic limit of the random perturbation in Section 5. The proof of Corollary 2.4 is presented at the end of Section 4, since it is there that all the necessary machinery is been developed.

Lemma 2.2. *In addition to* (2.1) *and* (2.2), *assume that the minimizing Mather measure is unique and $e \neq 0$ in* (2.12). *For any $C > 0$ and any $\varepsilon > 0$, there is a time $T_0 = T_0(C, \varepsilon) > 0$ such that, if γ is such that*

$$\int_0^T \left(L(\dot\gamma(t), \gamma(t)) + \overline{H}\right) dt \leq C \quad \textit{for all } T \geq T_0, \tag{2.20}$$

then

$$\left|\frac{\gamma(t) - \gamma(0)}{t} - e\right| \leq \varepsilon \quad \textit{for all } t \geq T_0.$$

We remark that the coercivity assumption and (2.20) imply that $\|\dot\gamma\|_{L^2}$ and, hence, $|\gamma(t)-\gamma(s)|$ are uniformly bounded on bounded (time) intervals of size less than T_0. The lemma above, provides a bound on $|\gamma(t)-\gamma(s)|$ for time intervals of length larger than T_0.

An immediate consequence of Lemma 2.2, which is used in the analysis of the asymptotic limits, is stated in the next corollary. Its proof, which is essentially a restatement of the conclusion of Lemma 2.2, is omitted.

Corollary 2.3. *In addition to* (2.1) *and* (2.2), *assume that the minimizing Mather measure is unique and* $e \neq 0$ *in* (2.12). *For any* $C, \theta > 0$, *there exist* $T_0 = T_0(C,\theta) > 0$ *and* $R_0 = R_0(C) > 0$, *such that, if* γ *satisfies the bound* (2.20) *with the given* C, *then*

$$\inf_{s\geq t}\langle \gamma(s)-\gamma(t), e\rangle \geq -R_0 \quad \text{and} \quad \inf_{s\geq t+T_0}\langle \gamma(s)-\gamma(t), e\rangle \geq \theta \textit{ for all } t\geq 0.$$

In particular, there exists $K = K(C) > 0$ *such that any* γ *satisfying* $\langle e, \gamma(0)\rangle \geq K$ *and* (2.20) *avoids the support of* ζ *for any positive time.*

The proof of Lemma 2.2. Let γ_n be a sequence of trajectories satisfying (2.20). In view of the periodicity, we may assume without loss of generality that $\gamma_n(0) \in Q$.

Let $\mu_{n,T}$ be the occupational measure on $\mathbb{R}^d \times \mathbb{R}^d$ which is periodic in space and defined, for all $\phi = \phi(\xi, x) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ which are periodic with respect to x, by

$$\int_{\mathbb{R}^d\times Q} \phi(\xi,x)d\mu_{n,T}(\xi,x) := \frac{1}{T}\int_0^T \phi(\dot\gamma_n(t),\gamma_n(t))dt.$$

It follows from (2.20) that

$$\int_{\mathbb{R}^d\times Q} L(\xi,x)d\mu_{n,T}(\xi,x) \leq -\overline{H} + \frac{C}{T},$$

and, hence, in view of the coercivity of L, the family $(\mu_{n,T})_{n\in\mathbb{N},T\geq 0}$ is tight.

Then, as n and $T\to\infty$, there exists a subsequence of $\mu_{n,T}$ (for simplicity we do not change the notation of the subsequence) that converges weakly to a measure μ satisfying

$$\int_{\mathbb{R}^d\times Q} L(\xi,x)d\mu(\xi,x) \leq -\overline{H}.$$

Note also that, for any $\mathbb{Z}^d$-periodic $\phi \in C^1(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d\times Q} \langle D\phi(x),\xi\rangle d\mu(\xi,x) = & \lim_{n,T\to\infty}\int_{\mathbb{R}^d\times Q}\langle D\phi(x),\xi\rangle \mu_{n,T}(\xi,x)\\ = & \lim_{n,T\to\infty}\frac{\phi(\gamma_n(T))-\phi(\gamma_n(0))}{T} = 0,\end{aligned}$$

that is μ is also closed, and, hence, a Mather minimizing measure and, in view of the assumed uniqueness, the entire family $(\mu_{n,T})_{n\in\mathbb{N},T\geq 0}$ converges to μ as $n, T\to\infty$. Moreover, μ is the image of $\widetilde\sigma$ by the map $x \mapsto (-D_pH(D\chi(x),x),x)$.

In particular, as n and $T\to\infty$,

$$\frac{\gamma_n(T)-\gamma_n(0)}{T} = \frac{1}{T}\int_0^T \dot\gamma_n(t)dt = \int_{\mathbb{R}^d\times Q}\xi d\mu_{n,T}(\xi,x) \to \int_Q -D_pH(D\chi(x),x)d\widetilde\sigma(x) = e.$$

The claim then follows from the assumption that $e \neq 0$. □

Finally the proof of Theorem 4.1 yields the following partial description of χ_∞ which is of independent interest.

Corollary 2.4. *Let $\mathcal{O}$ be the open set of points such that $(\overline{\gamma}^x)_{t\in\mathbb{R}}$ does not intersects* $\operatorname{sppt}(\zeta)$. *There exists $c \in \mathbb{R}$ such that*

(i) $\chi_\infty = \chi + c$ *in* $\mathcal{O}$,
(ii) $\chi_\infty \geq \chi + c$ *in* $\mathbb{R}^d$,
(iii) *there exists $K \geq 0$ such that, if $\langle x, e\rangle \geq K$, then $\chi_\infty(x) = \chi(x) + c$,*
(iv) *for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that, if $\langle x, e\rangle \leq -K_\varepsilon$, then $(\chi_\infty - \chi)(x) \leq c + \varepsilon$.*

The proof is presented at the end of Section 4.

3. The growth of the perturbed ergodic constant

Given a Hamiltonian H we consider perturbations of the form $H(p, x) - f(x)$, where f is a non negative potential and prove that, under assumptions on the potential, the difference of the corresponding effective constants can be controlled by some "average" of f.

We present two results, one for almost periodic and one for random media. Then we describe the relationship with the two examples in the introduction and prove (1.10) and (1.11).

Almost periodic perturbations. Let

$$f : \mathbb{R}^d \to \mathbb{R} \text{ be nonnegative, bounded uniformly continuous and almost periodic,} \tag{3.1}$$

and recall that almost periodicity implies the existence of the average

$$\fint_{\mathbb{R}^d} f = \lim_{R\to+\infty} \fint_{Q_R} f.$$

Given H satisfying (2.1) and (2.2), we consider the perturbed Hamiltonian

$$H_f(p, x) := H(p, x) - f(x),$$

and note that, for any $p \in \mathbb{R}^d$, $x \to H_f(p, x)$ is almost periodic.

We recall (see Ishii [16]) that the ergodic constant $\overline{H}_f$ associated with H_f is obtained as the uniform in $\mathbb{R}^d$ limit, as $\delta \to 0$, of $-\delta v^\delta$, where v^δ is the the unique bounded viscosity solution to

$$\delta v^\delta + H_f(Dv^\delta, x) = 0 \text{ in } \mathbb{R}^d.$$

It is straightforward implication of the comparison principle of viscosity solutions that

$$0 \leq \overline{H} - \overline{H}_f \leq \|f\|_\infty.$$

This estimate does not depend on the almost periodicity of f and, hence, is not useful here since it does not "see" the averaging that is taking place.

To obtain a more precise estimate of the difference $\overline{H} - \overline{H}_f$,we introduce the auxiliary quantity

$$\widehat{f}(x) := \limsup_{N\to+\infty} N^{-d} \sum_{Q_1(k)\subset Q_N} f(x + k),$$

and note that, since f is uniformly continuous, $\widehat{f}$ is a $\mathbb{Z}^d-$periodic and continuous map, and, moreover, and this is very important, $\|\widehat{f}\|_\infty$ is in general much smaller than $\|f\|_\infty$.

To illustrate the difference between $\|\widehat{f}\|_\infty$ and $\|f\|_\infty$, we discuss the example we considered in the introduction, that is the periodic perturbation ζ_R given by (1.4), which is obviously almost periodic, and we estimate $\|\widehat{\zeta}_R\|_\infty$.

Let $K > 0$ such that the support of ζ is contained in Q_K. Then, for any $x \in Q$ and $N \in \mathbb{N}$, we have

$$N^{-d} \sum_{Q_1(k) \subset Q_N} \zeta_R(x+k) \le N^{-d} \sum_{Q_1(k) \subset Q_N} \sum_{k' \in \mathbb{Z}^d} \|\zeta\|_\infty \mathbf{1}_{\{|k+Rk'| \le K+1\}}.$$

It follows that, for N sufficiently larger than R,

$$N^{-d} \sum_{Q_1(k) \subset Q_N} \zeta_R(x+k) \le N^{-d} \|\zeta\|_\infty \sum_{k' \in \mathbb{Z}^d} \sharp \left\{ k \in \mathbb{Z}^d : Q_1(k) \subset Q_N \cap Q_{K+1}(-Rk') \right\},$$

and, hence,

$$\|\widehat{\zeta}_R\|_\infty \lesssim R^{-d}.$$

Note that, as $R \to +\infty$, $\widehat{\zeta}_R \to 0$, whereas $\|\zeta_R\|_\infty = \|\zeta\|_\infty$ is constant.

The general result about the size of the perturbation is the following Theorem.

Theorem 3.1. *Assume* (2.1), (2.2) *and* (3.1). *Then*

$$0 \le \overline{H} - \overline{H}_f \le \|\widehat{f}\|_\infty.$$

In view of the computations above, we have the following corollary.

Corollary 3.2. *Assume* (2.1)*and* (2.2) *and consider, for $R \in \mathbb{N}$, the perturbation ζ_R given by* (1.4). *Then*

$$0 \le \overline{H} - \overline{H}_R \lesssim R^{-d}.$$

Theorem 3.1 states that $\overline{H}_f$ is close to $\overline{H}$ if the almost periodic perturbation f is nonnegative and small in "average". In the two examples discussed below we show that both conditions are sharp.

The assumption that ζ is nonnegative cannot be removed. we show this in the framework of Corollary 3.2. Indeed, if $H(p) = |p|^2$ and $H_R(p,x) = |p|^2 - \zeta_R(x)$, then it is known (see [19]) that $\overline{H} = 0$ and, independently of the sign of ζ_R, $\overline{H}_R = -\inf \zeta_R$. In particular, $\overline{H}_R$ does not converge to $\overline{H} = 0$ as $R \to +\infty$, if ζ takes negative values, since, in this case $\inf \zeta_R = \inf \zeta < 0$.

Next we discuss the manner in which the perturbation is averaged. When f is nonnegative, it seems reasonable to expect that $\|\widehat{f}\|_\infty$ can be replaced by the average of f in Proposition 3.1. This is, however, also not true. For example, fix $\varepsilon > 0$ small and $R > 0$ large and consider the perturbation

$$\zeta_R(x) = \varepsilon \sum_{k \in \mathbb{Z}^d} \zeta(R(x-k)),$$

with ζ satisfying (1.5) and $\zeta(0) = 1$; note that this $\mathbb{Z}^d-$periodic perturbation differs from the one in (1.4) and thus Corollary 3.2 does not apply here. Then, uniformly on $\varepsilon \in (0,1)$,

$$\lim_{R \to \infty} ⨍_{\mathbb{R}^d} \zeta_R = 0.$$

Moreover, if $H(p,x) = |p|^2 - g(x)$, where g is continuous, then $\overline{H} = -\inf g$. If, in addition, g has a unique and strict global minimum on Q at 0 and ε is sufficiently small independently of R, then $\inf(g + \zeta_R) \approx \inf g + \zeta_R(0) = \inf g + \varepsilon$. Hence $\overline{H}_{\zeta_R} \approx -(\inf g + \varepsilon)$ does not tend, as $R \to +\infty$, to $\overline{H} = -\inf g$.

Proof of Theorem 3.1. Since f is nonnegative, the comparison argument yields $\overline{H}_f \le \overline{H}$.

For the upper bound, it is convenient to regularize the problem and to consider, for $\varepsilon > 0$, the almost periodic solution $v^{\delta,\varepsilon}$ of the approximate viscous cell-problem

$$\delta v^{\delta,\varepsilon} - \varepsilon \Delta v^{\delta,\varepsilon} + H(Dv^{\delta,\varepsilon}, x) = f \text{ in } \mathbb{R}^d. \tag{3.2}$$

We recall that, as $\varepsilon \to 0$, $\delta v^{\delta,\varepsilon} \to \delta v^{\delta}$ uniformly in x and δ, where v^{δ} is the solution to

$$\delta v^{\delta} + H(Dv^{\delta}, x) = f \text{ in } \mathbb{R}^d.$$

We also consider the solution χ^{ε} of the viscous periodic cell-problem associated to H, that is

$$-\varepsilon \Delta \chi^{\varepsilon} + H(D\chi^{\varepsilon}, x) = \overline{H}^{\varepsilon} \text{ in } \mathbb{R}^d, \tag{3.3}$$

as well as the associated ergodic measure which has a continuous, strictly positive, $\mathbb{Z}^d-$periodic density $\widetilde{\sigma}^{\varepsilon}$ of mass 1 over Q satisfying

$$-\varepsilon \Delta \widetilde{\sigma}^{\varepsilon} - \operatorname{div}(\widetilde{\sigma}^{\varepsilon} D_p H(D\chi^{\varepsilon}, x)) = 0 \text{ in } \mathbb{R}^d. \tag{3.4}$$

Finally, we recall that $\lim_{\varepsilon \to 0} \overline{H}^{\varepsilon} = \overline{H}$, while the measure $\widetilde{\sigma}^{\varepsilon}$ converges, up to subsequences, to some Mather minimizing measure $\widetilde{\sigma}$.

Subtracting (3.3) from (3.2) and using the convexity of H we find that $v^{\delta,\varepsilon} - \chi^{\varepsilon}$ solves

$$-\varepsilon \Delta (v^{\delta,\varepsilon} - \chi^{\varepsilon}) + \delta v^{\delta,\varepsilon} + \langle D_p H(D\chi^{\varepsilon}, x), D(v^{\delta,\varepsilon} - \chi^{\varepsilon}) \rangle \le f - \overline{H}^{\varepsilon} \text{ in } \mathbb{R}^d.$$

Multiplying the above inequality by $\widetilde{\sigma}^{\varepsilon}$, integrating over Q_N for a large $N \in \mathbb{N}$ and using that σ^{ε} is an invariant measure, that is (3.4), we find

$$\begin{aligned} \delta \int_{Q_N} v^{\delta,\varepsilon} \widetilde{\sigma}^{\varepsilon} + \int_{\partial Q_N} \langle -\varepsilon D(v^{\delta,\varepsilon} - \chi^{\varepsilon}) + (v^{\delta,\varepsilon} - \chi^{\varepsilon}) D_p H(D\chi^{\varepsilon}, x)), \nu \rangle \widetilde{\sigma}^{\varepsilon} \\ \le \int_{Q_N} f(x) \widetilde{\sigma}^{\varepsilon}(x) dx - \overline{H}^{\varepsilon} N^d, \end{aligned} \tag{3.5}$$

where ν is the outward unit normal at Q_N.

Note that the periodicity of $\widetilde{\sigma}_{\epsilon}$ yields

$$\int_{Q_N} f(x) \widetilde{\sigma}^{\varepsilon}(x) dx = \int_Q \sum_{Q_1(k) \subset Q_N} f(x+k) \widetilde{\sigma}^{\varepsilon}(x) dx,$$

while the integrand of the integral over ∂Q_N in (3.5) is bounded.

Dividing (3.5) by N^d, letting $N \to +\infty$ and using Fatou's Lemma, we get

$$\delta \fint_{\mathbb{R}^d} v^{\delta,\varepsilon} \widetilde{\sigma}^{\varepsilon} \le \int_Q \widehat{f}(x) \widetilde{\sigma}^{\varepsilon}(x) dx - \overline{H}^{\varepsilon}.$$

Finally letting first $\varepsilon \to 0$ and then $\delta \to 0$ yields the claim, in view of the uniform convergence in $\mathbb{R}^d$ of $v^{\delta,\varepsilon}$ to v^{δ} (as $\varepsilon \to 0$) and of δv^{δ} to $-\overline{H}_f$ (as $\delta \to 0$) and by the convergence of $\widetilde{\sigma}^{\varepsilon}$ to $\widetilde{\sigma}$ in measure (as $\varepsilon \to 0$). $\square$

Random perturbations. We consider here a perturbation of the Hamiltonian H by a random potential f in a probability space described in the introduction.
We assume that

$$\begin{cases} f : \mathbb{R}^d \times \Omega \to \mathbb{R} \text{ is nonnegative, continuous with respect to the first variable} \\ \text{uniformly with respect to the second variable, and } \mathbb{Z}^d\text{−stationary,} \end{cases} \tag{3.6}$$

that is $f(x + k, \omega) = f(x, \tau_k \omega)$ for all $x \in \mathbb{R}^d, k \in \mathbb{Z}^d$ and $\omega \in \Omega$. It follows that the map $x \to \mathbb{E}[f(x, \cdot)]$ is a continuous and $\mathbb{Z}^d$−periodic function.
Let $\overline{H}_f$ be the ergodic constant associated with the Hamiltonian $H(p, x) - f(x)$ which exists (see [28]) and is identified by the a.s. limit $\overline{H}_f := \lim_{\delta \to 0} -\delta v^\delta(0)$, v^δ being the bounded and Lipschitz continuous, with a constant independent of δ, $\mathbb{Z}^d$-stationary solution to the discounted problem

$$\delta v^\delta + H(Dv^\delta, x) - f = 0 \text{ in } \mathbb{R}^d.$$

Theorem 3.3. *Assume* (2.1)*,* (2.2) *and* (3.6)*. Then*

$$0 \le \overline{H} - \overline{H}_f \le \sup_{x \in \mathbb{R}^d} \mathbb{E}[f(x, \cdot)].$$

We describe now the particular case of the above result which was discussed in the introduction and will be further investigated in Section 5.
Fix ζ satisfying (1.5) and, for $\eta \in (0, 1)$, let $(X_k^\eta)_{k \in \mathbb{Z}^d}$ be a family of i.i.d. Bernoulli random variables of parameter η, that is

$$\mathbb{P}[X_0^\eta = 1] = 1 - \mathbb{P}[X_0^\eta = 0] = \eta.$$

Set

$$\zeta_\eta(x) := \sum_{k \in \mathbb{Z}^d} X_k^\eta \zeta(x - k) \qquad \text{and} \qquad H_\eta(p, x) := H(p, x) - \zeta_\eta(x),$$

and denote by $\overline{H}_\eta$ (instead of $\overline{H}_{\zeta_\eta}$) the effective constant.
In this context, Theorm 3.3 yields immediately the following result.

Corollary 3.4. *Assume* (2.1)*,* (2.2) *and let ζ_η be defined as above. Then, for all $\eta \in (0, 1)$,*

$$0 \le \overline{H} - \overline{H}_\eta \lesssim \eta.$$

We continue with the:

Proof of Theorem 3.3. Using the notation and strategy of the proof of Theorem 3.1, we have

$$\begin{aligned} \delta \int_{Q_N} v^{\delta,\varepsilon} \widetilde{\sigma}^\varepsilon + \int_{\partial Q_N} \langle -D(v^{\delta,\varepsilon} - \chi_\varepsilon) + (v^{\delta,\varepsilon} - \chi^\varepsilon) D_p H(D\chi^\varepsilon, x)), \nu \rangle \widetilde{\sigma}^\varepsilon \\ \le \int_{Q_N} f(x, \omega) \widetilde{\sigma}^\varepsilon(x) dx - \overline{H}^\varepsilon N^d. \end{aligned}$$

Note that the maps $x \mapsto \mathbb{E}[v^{\delta,\varepsilon}(x)]$ and $x \mapsto \mathbb{E}[f(x)]$ are $\mathbb{Z}^d$−periodic. Hence taking expectation, dividing by N^d and letting $N \to +\infty$ in the above inequality, we find

$$\int_Q \delta \mathbb{E}[v^{\delta,\varepsilon}] \widetilde{\sigma}^\varepsilon \le \int_Q \mathbb{E}[f] \widetilde{\sigma}^\varepsilon - \overline{H}^\varepsilon,$$

and, letting first $\varepsilon \to 0$ and then $\delta \to 0$, we obtain

$$\overline{H} - \overline{H}_\eta \le \sup_x \mathbb{E}[f(x, \cdot)].$$

The inequality $\overline{H} \geq \overline{H}_\eta$ is an immediate consequence of the comparison principle. $\square$

4. Sharper convergence for the periodic perturbation

We revisit the periodic perturbation example we discussed in the introduction. Given H, the periodically perturbed Hamiltonian H_R is defined by (1.3) for some large $R \in \mathbb{N}$ with ζ_R as in (1.4) and ζ satisfying (1.5).

Let $\overline{H}$ and $\overline{H}_R$ be the effective constants associated with H and H_R respectively. In view of Corollary 3.2, we know that $R^d(\overline{H}_R - \overline{H})$ is bounded. Here we show that, under suitable assumptions on the unperturbed problem, this quantity has a limit.

The asymptotic result is stated next. Notice that claim depends nontrivially on the dimension.

Theorem 4.1. *Assume* (2.1), (2.2) *and* (2.6). *If* $d \geq 2$,

$$\lim_{R \to +\infty} R^d(\overline{H}_R - \overline{H}) = 0.$$

If $d = 1$ *and, in addition, the* $R\mathbb{Z}^d$*-periodic Hamiltonian* H_R *also satisfies* (2.6), *then*

$$\lim_{R \to +\infty} R(\overline{H}_R - \overline{H}) = -\Big(\int_Q \frac{1}{D_pH(\chi'(x), x)} dx\Big)^{-1} \int_{sppt(\zeta)} \big(H^{-1}(\zeta(x) + \overline{H}, x) - H^{-1}(\overline{H}, x)\big)\, dx.$$

The result when $d \geq 2$ is surprising. Indeed, as discussed in the introduction, the variational representations of χ_R and χ suggest that we should have $\overline{H} \approx \overline{H}_R - R^{-d} \int \zeta d\widetilde{\sigma}$ with $\int \zeta d\widetilde{\sigma}$ positive because $\widetilde{\sigma}$ has a full support. The claim in the Theorem 4.1 contradicts this intuition, since it implies that the optimal trajectories of the perturbed problem avoid the obstacles.

Of course, when $d = 1$ the optimal trajectories have no room to escape and need to go through the bumps.

We present first the proof of the result for $d = 1$, which is rather straightforward and is based on the exact formulae which are available in view of the assumptions. Then we move to the higher dimensional setting, which is more complicated and requires considerable more tools and work.

The problem in one dimension. We present here the:

Proof of Theorem 4.1 when $d = 1$. For $R \in \mathbb{N}$, we consider the cell problems (1.1) and (1.6). In view of the assumptions, the problems have $C^{1,1}$, with bounds independent of R, solutions χ and χ_R, which are respectively $\mathbb{Z}$- and $R\mathbb{Z}$- periodic.

Following the discussion in the subsection about the one-dimensional problem, we can also rewrite (1.6) as the ode

$$\chi_R' = H^{-1}(\zeta_R(x) + \overline{H}_R, x) \quad \text{in} \quad \mathbb{R}, \tag{4.1}$$

together with the condition

$$\int_{Q_R} H^{-1}(\zeta_R(x) + \overline{H}_R, x) dx = 0, \tag{4.2}$$

where $r \mapsto H^{-1}(r, x)$ is the same branch of the inverse of H we used for (2.13).

Let $S_R := \overline{H}_R - \overline{H}$ and recall that, in view of Corollary 3.2, $0 \leq R(-S_R) \leq C$.

We combine (2.14) and (4.2) as

$$\int_{Q_R} \big(H^{-1}(\zeta_R(x)+\overline{H}_R,x)-H^{-1}(\overline{H},x)\big)\,dx=0, \tag{4.3}$$

we rewrite it as

$$(\mathrm{I})_R+(\mathrm{II})_R=0, \tag{4.4}$$

with

$$(\mathrm{I})_R := \int_{Q_R} \big(H^{-1}(\zeta_R(x)+S_R+\overline{H},x)-H^{-1}(\zeta_R(x)+\overline{H},x)\big)\,dx,$$

and

$$(\mathrm{II})_R := \int_{Q_R} \big(H^{-1}(\zeta_R(x)+\overline{H},x)-H^{-1}(\overline{H},x)\big)\,dx,$$

and we study each term separately.

The strict convexity of H and the bound on S_R yield

$$|H^{-1}(\zeta_R(x)+S_R+\overline{H},x)-H^{-1}(\zeta_R(x)+\overline{H},x)-D_rH^{-1}(\zeta_R(x)+\overline{H},x)S_R|\lesssim(S_R)^2\lesssim R^{-2}$$

Then

$$|(\mathrm{I})_R-\frac{1}{R}\int_{Q_R} D_rH^{-1}(\zeta_R(x)+\overline{H},x)[RS_R]dx|\lesssim|Q_R||S_R|^2\lesssim R^{-1}.$$

Since R is large and ζ has compact support, in view of the definition of ζ_R, there is only one bump in Q_R. Hence, using (2.16), we get

$$\int_{Q_R} D_rH^{-1}(\zeta_R(x)+\overline{H},x)dx=\int_{Q_R}\big(D_rH^{-1}(\zeta(x)+\overline{H},x)-D_rH^{-1}(\overline{H},x)\big)\,dx+$$
$$\int_{Q_R} D_rH^{-1}(\overline{H},x)dx=R\int_Q D_rH^{-1}(\overline{H},x)dx+(\mathrm{III})_R=R(\int_Q\frac{1}{D_pH(\chi'(x),x)}dx)+(\mathrm{III})_R,$$

with

$$(\mathrm{III})_R := \int_{Q_R}\big(D_rH^{-1}(\zeta(x)+\overline{H},x)-D_rH^{-1}(\overline{H},x)\big)\,dx.$$

It is now immediate that

$$|(\mathrm{III})_R|=\left|\int_{\mathrm{sppt}(\zeta)}\big(D_rH^{-1}(\zeta(x)+\overline{H},x)-D_rH^{-1}(\overline{H},x)\big)\,dx\right|\lesssim 1.$$

Collecting all the previous information we find

$$\left|(\mathrm{I})_R-(\int_Q\frac{1}{D_pH(\chi'(x),x)}dx)RS_R\right|\lesssim R^{-1}. \tag{4.5}$$

Turning out attention to $(\mathrm{II})_R$ we observe that, since ζ has compact support,

$$(\mathrm{II})_R=\int_{\mathrm{sppt}(\zeta)}\big(H^{-1}(\zeta(x)+\overline{H},x)-H^{-1}(\overline{H},x)\big)\,dx.$$

The claim now follows. $\square$

The multi-dimensional problem. The main tool of the proof when $d \geq 2$ is the perturbed corrector χ_∞, that is a solution to (1.15), which, as it turns out, keeps tracks of the difference between $\overline{H}_R$ and $\overline{H}$; see Lemma 4.2 below. The core of the argument consists in showing that the difference $\chi_\infty - \chi$ tends to a constant at infinity. This statement relies heavily on the assumption that the projected invariant measure has a full support (see Lemma 4.4 and its proof).

Lemma 4.2. *For any $t > 0$, the map $x \mapsto [(\chi_\infty - \chi)(\overline{\gamma}^x(s)]_0^t$ belongs to $L^1_{\widetilde{\sigma}}(\mathbb{R}^d)$ and*

$$\liminf_{R\to+\infty} R^d(\overline{H}_R - \overline{H}) \geq t^{-1} \int_{\mathbb{R}^d} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^t \, d\widetilde{\sigma}(x).$$

Proof. The variational representation formulae for viscosity solutions to convex Hamilton-Jacobi equations give, for any $t > 0$,

$$\chi_R(x) = \inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot\gamma(s), \gamma(s)) + \overline{H}_R + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t)) \right],$$

and

$$\chi_\infty(x) = \inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot\gamma(s), \gamma(s)) + \overline{H} + \zeta(\gamma(s)))\, ds + \chi_\infty(\gamma(t)) \right], \tag{4.6}$$

where $\mathcal{A}_x$ is the set of Lipschitz curves $\gamma : [0, \infty) \to \mathbb{R}^d$ such that $\gamma(0) = x$, and

$$\chi(x) = \int_0^t (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \overline{H})\, ds + \chi(\overline{\gamma}^x(t));$$

recall that L is defined by (2.3), while the last equality is (2.11).
Hence, as in the introduction,

$$\begin{aligned} t(\overline{H}_R - \overline{H}) \;=\; & -\inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot\gamma(s), \gamma(s)) + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t)) \right] + \chi_R(x) \\ & + \int_0^t L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s))\, ds + \chi(\overline{\gamma}^x(t)) - \chi(x, \end{aligned}$$

and, after integrating over Q_R with respect to the measure $\widetilde{\sigma}$,

$$\begin{aligned} tR^d(\overline{H}_R - \overline{H}) \;=\; & \int_{Q_R} \Big\{ -\inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot\gamma(s), \gamma(s)) + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t)) \right] + \chi_R(x) \\ & + \int_0^t L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s))\, ds - \chi(\overline{\gamma}^x(t)) + \chi(x) \Big\}\, d\widetilde{\sigma}(x). \end{aligned}$$

Since the map $x \mapsto \overline{\gamma}^x(t)$ leaves the measure $\widetilde{\sigma}$ invariant (on the torus) and χ and χ_R are respectively $\mathbb{Z}^d$- and $R\mathbb{Z}^d$- periodic, for $R \in \mathbb{N}$, we have

$$\int_{Q_R} \chi(x)\, d\widetilde{\sigma}(x) = \int_{Q_R} \chi(\overline{\gamma}^x(t))\, d\widetilde{\sigma}(x) \quad \text{and} \quad \int_{Q_R} \chi_R(x)\, d\widetilde{\sigma}(x) = \int_{Q_R} \chi_R(\overline{\gamma}^x(t))\, d\widetilde{\sigma}(x).$$

Therefore

$$\begin{aligned} tR^d(\overline{H}_R - \overline{H}) \;=\; & -\int_{Q_R} \int_0^t \zeta_R(\overline{\gamma}^x(s))\, ds d\widetilde{\sigma}(x) \\ & + \int_{Q_R} \Big\{ -\inf_{\mathcal{A}_x} \left[\int_0^t (L(\dot\gamma(s), \gamma(s)) + \zeta_R(\gamma(s)))\, ds + \chi_R(\gamma(t)) \right] \\ & + \left[\int_0^t (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \zeta_R(\overline{\gamma}^x(s)))\, ds + \chi_R(\overline{\gamma}^x(t)) \right] \Big\}\, d\widetilde{\sigma}(x). \end{aligned}$$

We now discuss the behavior, as $R \to +\infty$, of the two integrals in the righthand side of the equality above.

Recall that $\dot{\overline{\gamma}}^x$ is uniformly bounded. Therefore, $\overline{\gamma}^x(s)$ does not see the bumps $\zeta(\cdot - kR)$ for $k \in \mathbb{Z}^d \backslash \{0\}$ as soon as $x \in Q_R$, $s \in [0,t]$ and R large enough with respect to t. Thus, for sufficiently large R,

$$\int_{Q_R} \int_0^t \zeta_R(\overline{\gamma}^x(s))\ dsd\widetilde{\sigma}(x) = \int_{\mathbb{R}^d} \int_0^t \zeta(\overline{\gamma}^x(s))\ dsd\widetilde{\sigma}(x).$$

For the second integral, we note that the integrand is nonnegative. Using Fatou's Lemma and the convergence of χ_R to χ_∞, we find

$$\begin{aligned} 0 \geq t \liminf_{R\to+\infty} R^d(\overline{H}_R - \overline{H})\ \geq\ & -\int_{\mathbb{R}^d} \int_0^t \zeta(\overline{\gamma}^x(s))\ dsd\widetilde{\sigma}(x) \\ & + \int_{\mathbb{R}^d} \Big\{ -\inf_{\mathcal{A}_x} \Big[\int_0^t (L(\dot{\gamma}(s), \gamma(s)) + \zeta(\gamma(s)))\ ds + \chi_\infty(\gamma(t)) \Big] \\ & \quad + \Big[\int_0^t (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \zeta(\overline{\gamma}^x(s)))\ ds + \chi_\infty(\overline{\gamma}^x(t)) \Big] \Big\}\ d\widetilde{\sigma}(x), \end{aligned}$$

which, in view of the nonnegativity of the integrand in the second integral, yields the $\widetilde{\sigma}$ integrability of the map

$$\begin{aligned} x \mapsto \xi(x) :=\ & -\inf_{\mathcal{A}_x} \Big[\int_0^t (L(\dot{\gamma}(s), \gamma(s)) + \zeta(\gamma(s)))\ ds + \chi_\infty(\gamma(t)) \Big] \\ & + \Big[\int_0^t (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \zeta(\overline{\gamma}^x(s)))\ ds + \chi_\infty(\overline{\gamma}^x(t)) \Big]. \end{aligned}$$

It follows from (4.6) and (2.11) that

$$\begin{aligned} \xi(x)\ &=\ -\big[\chi_\infty(x) - t\overline{H}\big] + \Big[\chi(x) - \chi(\overline{\gamma}^x(t)) - t\overline{H} + \int_0^t \zeta(\overline{\gamma}^x(s))\ ds + \chi_\infty(\overline{\gamma}^x(t)) \Big] \\ &=\ \big[(\chi_\infty - \chi)(\overline{\gamma}^x(s))\big]_0^t + \int_0^t \zeta(\overline{\gamma}^x(s))\ ds. \end{aligned}$$

Since the map $x \mapsto \int_0^t \zeta(\overline{\gamma}^x(s))\ ds$ has compact support, it is integrable with respect to $\widetilde{\sigma}$. This last observation together with the integrability of ξ yield the first assertion. The second one is immediate from the formulae above. □

The next lemma is about the fact that, at least along the optimal trajectories, $\chi_\infty - \chi$ has limits.

Lemma 4.3. *For $\widetilde{\sigma}$−a.e. $x \in \mathbb{R}^d$, the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non decreasing on any time-interval $[t_1, t_2]$ such that $\overline{\gamma}^x$ does not encounter* $\mathrm{sppt}(\zeta)$. *In particular, the limits*

$$c^\pm(x) := \lim_{t\to\pm\infty} (\chi_\infty - \chi)(\overline{\gamma}^x(t))$$

exist for $\widetilde{\sigma}$−a.e. $x \in \mathbb{R}^d$ and, if $\overline{\gamma}^x$ never encounters $\mathrm{sppt}(\zeta)$, *then $c^-(x) \leq c^+(x)$.*

Proof. Let $x \in \mathbb{R}^d$, and $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $\gamma^x([t_1,t_2]) \cap \mathrm{sppt}(\zeta) = \emptyset$, and, hence, $\zeta(\overline{\gamma}^x) = 0$ on $[t_1, t_2]$. Using (4.6) and (2.11), for any $t_1 \leq s_1 \leq s_2 \leq t_2$, we find

$$\begin{aligned} \chi_\infty(\overline{\gamma}^x(s_1))\ &\leq\ \int_{s_1}^{s_2} (L(\dot{\overline{\gamma}}^x(s), \overline{\gamma}^x(s)) + \overline{H} + \zeta(\overline{\gamma}^x(s)))\ ds + \chi_\infty(\overline{\gamma}^x(s_2)) \\ &=\ \chi(\overline{\gamma}^x(s_1)) - \chi(\overline{\gamma}^x(s_2)) + \chi_\infty(\overline{\gamma}^x(s_2)). \end{aligned}$$

Since, for $\widetilde{\sigma}-$a.e. $x \in \mathbb{R}^d$,

$$\lim_{t\to\pm\infty} \frac{\overline{\gamma}^x(t)}{t} = e \neq 0,$$

there exists T such that $\overline{\gamma}^x$ does not intersect $\mathrm{sppt}(\zeta)$ for $|t| \geq T$. Then $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non decreasing and bounded on the intervals $(-\infty, -T]$ and $[T, +\infty)$, and the claimed limits exist.

If $\overline{\gamma}^x$ does not encounter $\mathrm{sppt}(\zeta)$ at all, then $t \to (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non decreasing on $\mathbb{R}$, and, hence,

$$c^-(x) \leq \lim_{t\to-\infty}(\chi_\infty - \chi)(\overline{\gamma}^x(t)) \leq \lim_{t\to+\infty}(\chi_\infty - \chi)(\overline{\gamma}^x(t)) = c^+(x).$$

□

The next step is crucial, since it asserts that it is possible to always compare c^- and c^+.

Lemma 4.4. *The maps c^+ and c^- are constant with $c^- \leq c^+$.*

Proof. Fix $x_0 \in \mathbb{R}^d$ such that (2.12) holds. Since $\widetilde{\sigma}$ is ergodic, there exist $t_n \to +\infty$ and $k_n \in \mathbb{Z}^d$ such that, as $n \to \infty$,

$$|\overline{\gamma}^{x_0}(t_n) - x_0 - k_n| \to 0.$$

In view of the $\mathbb{Z}^d$-periodicity of the flow $x \mapsto \overline{\gamma}^x$, that is the fact that, for every $k \in \mathbb{Z}^d$, $\overline{\gamma}^{x+k} = \overline{\gamma}^x + k$, we also have, for all $k \in \mathbb{Z}^d$,

$$\lim_{t\to\pm\infty} \frac{\overline{\gamma}^{x_0+k}(t)}{t} = e \neq 0, \tag{4.7}$$

and

$$\lim_{n\to\infty} \left|\overline{\gamma}^{x_0+k}(t_n) - x_0 - k - k_n\right| = 0. \tag{4.8}$$

The boundedness and Lipschitz continuity of χ_∞ allow to choose a sequence $(\chi_\infty(\cdot + k_n))_{n\in\mathbb{N}}$ that converges locally uniformly to some map χ_∞^+.

Fix now some $k \in \mathbb{Z}^d$. Then (4.7) yields some t_0 such that $\overline{\gamma}^{x_0+k}([t_0,\infty)) \cap \mathrm{sppt}(\zeta) = \emptyset$, and, in turn, Lemma 4.3 states that the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^{x_0+k}(t))$ is nondecreasing and converges to $c^+(x_0 + k)$ as $t \to +\infty$.

In particular, for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty}(\chi_\infty - \chi)(\overline{\gamma}^{x_0+k}(t_n)) = \lim_{n\to\infty}(\chi_\infty - \chi)(\overline{\gamma}^{x_0+k}(t_n + t)) = c^+(x_0 + k).$$

Then (4.8) and the continuity and periodicity of χ give

$$\lim_n \chi(\overline{\gamma}^{x_0+k}(t_n)) = \lim_n \chi(x_0 + k + k_n) = \chi(x_0 + k),$$

while the uniform continuity of χ_∞ implies

$$\lim_{n\to\infty} \chi_\infty(\overline{\gamma}^{x_0+k}(t_n)) = \lim_{n\to\infty} \chi_\infty(x_0 + k + k_n) = \chi_\infty^+(x_0 + k).$$

Note also that, using (4.8) with $k = 0$, and the periodicity and continuity of flow $x \mapsto \overline{\gamma}^x$, we find that, as $n \to \infty$,

$$\left|\overline{\gamma}^{x_0+k}(t_n + t) - \overline{\gamma}^{x_0+k}(t) - k_n\right| = \left|\overline{\gamma}^{\overline{\gamma}^{x_0}(t_n)-k_n}(t) - \overline{\gamma}^{x_0}(t)\right| \to 0.$$

In conclusion

$$\begin{aligned} c^+(x_0+k) &= \lim_{n\to\infty}(\chi_\infty-\chi)(\overline{\gamma}^{x_0+k}(t_n+t)) \\ &= \lim_{n\to\infty}(\chi_\infty-\chi)(\overline{\gamma}^{x_0+k}(t)+k_n) = (\chi_\infty^+-\chi)(\overline{\gamma}^{x_0+k}(t)). \end{aligned}$$

This proves that, for all $t\in\mathbb{R}$ and $k\in\mathbb{Z}^d$,

$$(\chi_\infty^+-\chi)(x_0+k) = (\chi_\infty^+-\chi)(\overline{\gamma}^{x_0+k}(t)) = c^+(x_0+k). \tag{4.9}$$

Next we show that the map $t\mapsto(\chi_\infty^+-\chi)(\overline{\gamma}^x(t))$ is constant for any $x\in\text{sppt}(\widetilde{\sigma})$.

Fix $x\in\text{sppt}(\widetilde{\sigma})$. Since $\widetilde{\sigma}$ is ergodic with full support, there exist sequences $s_n\to+\infty$ and $m_n\in\mathbb{Z}^d$ such that, as $n\to\infty$,

$$x-\overline{\gamma}^{x_0}(s_n)-m_n = x-\overline{\gamma}^{x_0+m_n}(s_n)\to 0.$$

Then (4.9) implies that the map

$$s\mapsto(\chi_\infty^+-\chi)(\overline{\gamma}^{x_0+m_n}(s_n+s)) = (\chi_\infty^+-\chi)(\overline{\gamma}^{\overline{\gamma}^{x_0+m_n}(s_n)}(s))$$

has constant on $\mathbb{R}$ value $c^+(x_0+m_n)$ and converges locally uniformly to $s\to(\chi_\infty^+-\chi)(\overline{\gamma}^x(s))$, which is therefore also constant on $\mathbb{R}$, that is, for all $s\in\mathbb{R}$ and all $x\in\text{sppt}(\widetilde{\sigma})$

$$\lim_{n\to\infty}c^+(x_0+m_n) = (\chi_\infty^+-\chi)(\overline{\gamma}^x(s)) = (\chi_\infty^+-\chi)(x). \tag{4.10}$$

Finally, (2.12) and the fact that $|k_n|\to+\infty$ imply that χ_∞^+ solves the same equation as χ, that is

$$H(D\chi_\infty^+,x)=\overline{H}\qquad\text{in }\mathbb{R}^d,$$

the main difference being that χ_∞^+ is not periodic a priori.

The next step is to show that $D\chi_\infty^+=D\chi$ a.e. and for this we use that $\widetilde{\sigma}>0$ a.e..

Let x be a point of differentiability of χ_∞^+ and note χ is differentiable at x. Then, since $\dot{\overline{\gamma}}^x(0)=-D_pH(x,D\chi(x))$, (4.10) gives

$$\langle D(\chi_\infty^+-\chi)(x),-D_pH(x,D\chi(x))\rangle=0.$$

On the other hand the uniform convexity of H implies, for some $C>0$,

$$\begin{aligned} 0 &= H(D\chi_\infty^+,x)-H(D\chi,x) \\ &\ge \langle D_pH(D\chi,x),D(\chi_\infty^+-\chi)(x)\rangle + C|D(\chi_\infty^+-\chi)(x)|^2 \\ &\ge C|D(\chi_\infty^+-\chi)(x)|^2. \end{aligned}$$

This proves that $D\chi_\infty^+=D\chi$ a.e. and, in particular, that $\chi_\infty^+-\chi$ is constant, which means that χ_∞^+ is periodic.

It follows from (4.9) that $c^+(x_0+k)=c^+(x_0)$ for any $k\in\mathbb{Z}^d$, which, using (4.10), leads to $c^+(x_0)=(\chi_\infty^+-\chi)(x)$ for any $x\in\mathbb{R}^d$.

A symmetric construction shows that c^- is also constant.

Finally, (2.12) yields some $k_0\in\mathbb{Z}^d$ large enough such that the trajectory $\overline{\gamma}^{x_0+k_0}$ does not encounter $\text{sppt}(\zeta)$ and, in view of Lemma 4.3, $c^-=c^-(x_0+k_0)\le c^+(x_0+k_0)=c^+$. Note that we use here the fact that we work in dimension $d\ge 2$, since otherwise it is not true that the trajectory $\overline{\gamma}^{x_0+k_0}$ does not intersect $\text{sppt}(\zeta)$ for k_0 large enough. $\square$

We continue with the:

Proof of Theorem 4.1 for $d \geq 2$. Fix $r \geq 1$ large enough so that Q_r contains the support of ζ and set $F := \{\overline{\gamma}^x(t) : x \in Q_r,\ t \in \mathbb{R}\}$, that is F contains all points that can be reached by optimal trajectories starting in Q_r at some time $t \in \mathbb{R}$.

The continuity of the flow $(t, x) \mapsto \overline{\gamma}^x(t)$ and (2.12) imply the existence a time $T_0 \in \mathbb{N}$ such that, for any $x \in Q_r$ and all t such that $|t| \geq T_0$,

$$\overline{\gamma}^x(t) \notin Q_r. \tag{4.11}$$

Since $\overline{H}_R \leq \overline{H}$, Lemma 4.2 suggests that to conclude, we just need to check that

$$I := \int_{\mathbb{R}^d} [(\chi_\infty - \chi)(\overline{\gamma}^x(s)]_0^{T_0}\, d\widetilde{\sigma}(x) \geq 0.$$

Lemma 4.3 states that $[(\chi_\infty - \chi)(\overline{\gamma}^x(s)]_0^{T_0}$ is nonnegative, if the trajectory $\overline{\gamma}^x$ does not encounter $\mathrm{sppt}(\zeta)$. Since $\mathrm{sppt}(\zeta) \subset Q_r$, F in particular contains all the initial positions x such that $\overline{\gamma}^x$ intersects the support of ζ. Thus

$$I \geq \int_F [(\chi_\infty - \chi)(\overline{\gamma}^x(s)]_0^{T_0}\, d\widetilde{\sigma}(x).$$

The set F is not precise enough and we do not have much control over its size. In the next step, we replace it by a smaller one $\widetilde{F}$, which carries more information, without increasing by "too much" the size of the lower bound on I in the inequality above.

Let us recall that, from Lemma 4.3, $(\chi_\infty - \chi)(\overline{\gamma}^x(s))$ converges to $c^\pm$ as $\pm s \to +\infty$ for a.e. $x \in Q_r$. So, by Egoroff's theorem, for any fixed $\varepsilon > 0$, there exists a compact subset K of $Q_r \subset F$ and a time $T \in \mathbb{N}$ such that

(i) if $x \in K$ and $\pm s \geq T$, then $\left|(\chi_\infty - \chi)(\overline{\gamma}^x(s)) - c^\pm\right| \leq \varepsilon$,

(ii) if $\widetilde{F} := \{\overline{\gamma}^x(s) : x \in K,\ s \in \mathbb{R}\}$, then

$$\int_F [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0}\, d\widetilde{\sigma}(x) \geq \int_{\widetilde{F}} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0}\, d\widetilde{\sigma}(x) - \varepsilon$$

and

$$\widetilde{\sigma}(\widetilde{F} \cap Q_r) \geq \widetilde{\sigma}(F \cap Q_r) - \varepsilon = \widetilde{\sigma}(Q_r) - \varepsilon.$$

The next step is to provide a more precise characterization for $\widetilde{F}$. For this we construct $E \subset K$ such that

$$\widetilde{F} = \{\overline{\gamma}^x(t) : x \in E,\ t \in \mathbb{R}\} \text{ and } \{\overline{\gamma}^x(t) : t \in \mathbb{R}\} \cap E = \{x\} \text{for all } x \in E. \tag{4.12}$$

For any $x \in K$, let $\tau_x := \inf\{t \in \mathbb{R} : \overline{\gamma}^x(t) \in K\}$ and set $E := \{\overline{\gamma}^x(\tau_x) : x \in Q_r\}$. It is immediate that E is a Borel measurable set and satisfies (4.12).

For $k \in \mathbb{Z}$, set

$$E(k) := \{\overline{\gamma}^x(t) : t \in [k, k+1)T_0,\ x \in E\}.$$

The family $(E(k))_{k \in \mathbb{Z}}$ is a partition of $\widetilde{F}$. Moreover note that the definition of E and (4.11) yield

$$\widetilde{F} \cap Q_r \subset E(0).$$

For $n \in \mathbb{N}$, $n \geq T$ and large enough, we have

$$\begin{aligned}
&\int_{\widetilde{F}} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0} \; d\widetilde{\sigma}(x) \\
&\quad \geq \sum_{k=-n-1}^{n} \int_{E(k)} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0} \; d\widetilde{\sigma}(x) - \varepsilon \\
&\quad = \sum_{k=-n-1}^{n} \left[\int_{E(k)} (\chi_\infty - \chi)(\overline{\gamma}^x(T_0)) \; d\widetilde{\sigma}(x) - \int_{E(k)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) \right] - \varepsilon \\
&\quad = \sum_{k=-n-1}^{n} \left[\int_{E(k+1)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) - \int_{E(k)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) \right] - \varepsilon,
\end{aligned}$$

where we used that $\widetilde{\sigma}$ is invariant by the flow $\overline{\gamma}^x$ and the image by the map $x \mapsto \overline{\gamma}^x(T_0)$ of $E(k)$ is $E(k+1)$.

Hence

$$\begin{aligned}
&\int_{\widetilde{F}} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0} \; d\widetilde{\sigma}(x) \\
&\quad \geq \int_{E(T+1)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) - \int_{E(-T-1)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) - \varepsilon.
\end{aligned}$$

The choice of $\widetilde{F}$ (property (i)) gives

$$\begin{aligned}
\int_{E(T+1)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) &= \int_{E(0)} (\chi_\infty - \chi)(\overline{\gamma}^x(T+1)) \; d\widetilde{\sigma}(x) \\
&\geq (c^+ - \varepsilon)\widetilde{\sigma}(E(0)),
\end{aligned}$$

and, similarly,

$$\int_{E(-T-1)} (\chi_\infty - \chi)(x) \; d\widetilde{\sigma}(x) \leq (c^- + \varepsilon)\widetilde{\sigma}(E(0)).$$

Thus, since $c^+ \geq c^-$ and $\widetilde{F} \cap Q_r \subset E(0)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0} \; d\widetilde{\sigma}(x) \\
&\quad \geq (c^+ - c^-)\widetilde{\sigma}(E(0)) - C\varepsilon \; \geq \; (c^+ - c^-)\widetilde{\sigma}(\widetilde{F} \cap Q_r) - C\varepsilon.
\end{aligned}$$

Using property (ii) in the definition of $\widetilde{F}$, we finally find

$$I = \int_{\mathbb{R}^d} [(\chi_\infty - \chi)(\overline{\gamma}^x(s))]_0^{T_0} \; d\widetilde{\sigma}(x) \geq (c^+ - c^-)\widetilde{\sigma}(Q_r) - C\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$0 \geq I \geq (c^+ - c^-)\widetilde{\sigma}(Q_r) \geq 0,$$

which yields both $I = 0$ and $c^+ = c^-$. □

The proof of Corollary 2.4. Let c be the common value of $c^\pm$; recall that in the proof of Theorem 4.1 we showed that $c^+ = c^-$. Let now $x \in \mathcal{O}$, the open set of points such that $(\overline{\gamma}^x(t))_{t \in \mathbb{R}}$ does not intersects $\mathrm{sppt}(\zeta)$. Lemma 4.3 implies that the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non increasing and has the same limit c at $-\infty$ and $+\infty$. Hence it is constant in $\mathcal{O}$ and (i) holds.

Next we show that, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$(\chi_\infty - \chi)(\overline{\gamma}^x(t)) \geq c - \varepsilon \;\; \text{for all } x \in \mathrm{sppt}(\zeta) \text{ and } t \geq T_\varepsilon.$$

Fix $x \in \mathrm{sppt}(\zeta)$. We know from Lemma 4.3 that the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non decreasing for t large enough and converges to c. Hence, there exists T_x such that $|(\chi_\infty - \chi)(\overline{\gamma}^x(t)) - c| \le \varepsilon/2$ for $t \ge T_x$.

Sine the map $y \to \overline{\gamma}^y(t)$ is continuous, we also have $|(\chi_\infty - \chi)(\overline{\gamma}^y(T_x)) - c| \le \varepsilon$ for any y in some ball centered at x and of radius $\delta_x > 0$.

Thus, since the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^y(t))$ is nondecreasing, for $y \in B(x, \delta_x)$ and $t \ge T_x$ we have $(\chi_\infty - \chi)(\overline{\gamma}^y(t)) \ge c - \varepsilon$. We conclude using a standard compactness argument.

Next we claim that, for any $\varepsilon > 0$, there exists $K_\varepsilon \ge 0$ such that, for all $x \in \mathbb{R}^d$ such that $\langle x, e\rangle \le -K_\varepsilon$,

$$(\chi_\infty - \chi)(x) \ge c - \varepsilon. \tag{4.13}$$

Fix $\varepsilon > 0$, let T_ε be defined as in the previous step and choose

$$K_\varepsilon \ge M T_\varepsilon + \sup_{y \in \mathrm{sppt}(\zeta)} \langle y, e\rangle,$$

with $M := \|D_p H(D\chi, \cdot)\|_\infty$; notice that $\|\dot{\overline{\gamma}}^x\|_\infty \le M$.

It follows from Corollary 2.3 that, if K_ε is large enough, then for any $x \in \mathbb{R}^d$ with $\langle x, e\rangle \ge K_\varepsilon$, the trajectory $\overline{\gamma}^x(t)$ does not instersect $\mathrm{sppt}(\zeta)$ for $t \ge 0$.

Fix $x \in \mathbb{R}^d$ with $\langle x, e\rangle \ge K_\varepsilon$. If $\overline{\gamma}^x(\mathbb{R}) \cap \mathrm{sppt}(\zeta) = \emptyset$, we know that $(\chi_\infty - \chi)(x) = c$.

We now assume that there exists $t \in \mathbb{R}$ such that $\overline{\gamma}^x(t) \in \mathrm{sppt}(\zeta)$. Then, by the definition of T_ε, $t \le 0$, and, since as $\langle x, e\rangle \ge K_\varepsilon$ and $\overline{\gamma}^x(t) \in \mathrm{sppt}(\zeta)$, we must have

$$|x - \overline{\gamma}^x(t)| \ge \langle x - \overline{\gamma}^x(t), e\rangle \ge K_\varepsilon - \sup_{y \in \mathrm{sppt}(\zeta)} \langle y, e\rangle \;\ge\; \|\dot{\overline{\gamma}}^x\|_\infty T_\varepsilon,$$

and, hence, $|t| \ge T_\varepsilon$. But then the first claim of the corollary gives

$$(\chi_\infty - \chi)(x) = (\chi_\infty - \chi)(\overline{\gamma}^{\overline{\gamma}^x(t)}(-t)) \ge c - \varepsilon,$$

which proves (4.13).

A similar argument shows that, for all $\varepsilon > 0$, there exists $K_\varepsilon \ge 0$ such that, for all $x \in \mathbb{R}^d$ such that $\langle x, e\rangle \le -K_\varepsilon$,

$$(\chi_\infty - \chi)(x) \le c + \varepsilon, \tag{4.14}$$

which implies (iv).

We now prove (ii). Fix $x \in \mathbb{R}^d$ and let $\widetilde{\gamma}$ be the optimal path for $\chi_\infty(x)$ for positive times. Then

$$\chi_\infty(x) = \int_0^t \Big(L(\dot{\widetilde{\gamma}}(s), \widetilde{\gamma}(s)) + \zeta(\widetilde{\gamma}(s) + \overline{H} \Big)\, ds + \chi_\infty(\widetilde{\gamma}(t)), \tag{4.15}$$

while

$$\chi(x) \le \int_0^t \Big(L(\dot{\widetilde{\gamma}}(s), \widetilde{\gamma}(s)) + \overline{H} \Big)\, ds + \chi(\widetilde{\gamma}(t)). \tag{4.16}$$

Fix $\varepsilon > 0$ and let K_ε be given by the previous step. It follows from (4.15) that, for all $t \ge 0$,

$$\int_0^t \Big(L(\dot{\widetilde{\gamma}}(s), \widetilde{\gamma}(s)) + \overline{H} \Big)\, ds \le 2\|\chi_\infty\|_\infty,$$

which, in view of Lemma 2.2, implies that there exists $T_\varepsilon > 0$ such that $\langle \widetilde{\gamma}(t), e\rangle \ge K_\varepsilon$ for all $t \ge T_\varepsilon$. Then the definition of K_ε gives that, for all $t \ge T_\varepsilon$,

$$(\chi_\infty - \chi)(\widetilde{\gamma}(t)) \ge c - \varepsilon.$$

Subtracting (4.16) from (4.15) and using $\zeta \geq 0$ yields

$$(\chi_\infty - \chi)(x) \geq \int_0^{T_\varepsilon} \zeta(\widetilde{\gamma}(s))ds + (\chi_\infty - \chi)(\widetilde{\gamma}(T_\varepsilon)) \geq c - \varepsilon,$$

which proves (ii) since ε is arbitrary.

It remains to show (iii). Let K be given by Lemma 2.2 for $C := 2\|\chi_\infty\|_\infty$ and choose $x \in \mathbb{R}^d$ such that $\langle x, e\rangle \geq K$.

It follows, again from Lemma 2.2, that the trajectory $\overline{\gamma}^x$ avoids the support of ζ for all positive times. Then Lemma 4.3 implies that the map $t \mapsto (\chi_\infty - \chi)(\overline{\gamma}^x(t))$ is non decreasing on $[0, +\infty)$, so that

$$(\chi_\infty - \chi)(x) \ \leq \lim_{t\to+\infty} (\chi_\infty - \chi)(\overline{\gamma}^x(t)) = c.$$

Since, in view of (ii), the opposite inequality always holds, (iii) is proved. $\square$

5. Sharper convergence for the random perturbation

Given a $\mathbb{Z}^d$-periodic Hamiltonian satisfying (2.1), (2.2) and (2.6), we consider the random perturbation H_η defined in (1.7), with $\zeta : \mathbb{R}^d \to [0, +\infty)$ and $(X_k^\eta)_{k\in\mathbb{Z}^d}$ as in (1.5) and (1.9) respectively.

Let $\overline{H}_\eta$ be the effective constant associated with H_η. We are interested in the behavior of the ratio $(\overline{H}_\eta - \overline{H})/\eta$ as $\eta \to 0$; recall that in Corollary 3.4 we proved that this ratio is bounded.

We prove that $\lim_{\eta\to 0}(\overline{H}_\eta - \overline{H})/\eta$ exists and equals a non-zero constant when $d = 1$ and 0 when $d \geq 2$.

Theorem 5.1. *Assume* (2.1), (2.2), (2.6), (1.7), (1.5) *and* (1.9). *When* $d = 1$,

$$\lim_{\eta\to 0^+} \frac{\overline{H} - \overline{H}_\eta}{\eta} = -\Big(\int_Q \frac{1}{D_pH(D\chi, x)}dx\Big)^{-1} \int_{sppt(\zeta)} \big(H^{-1}(\zeta(x) + \overline{H}, x) - H^{-1}(\overline{H}, x)\big)\, dx,$$

and, when $d \geq 2$,

$$\lim_{\eta\to 0^+} \frac{\overline{H} - \overline{H}_\eta}{\eta} = 0.$$

A discussion similar to the one after Theorem 4.1 explains the difference between the one and the multi-dimensional case.

The proof for $d = 1$ makes strong use of the fact that in this case there exist "almost explicit formulae" for $\overline{H}$ and $\overline{H}_\eta$ and follows along the lines of the proof of the $d = 1$ limit in Theorem 4.1.

Proof of Theorem 5.1 for $d = 1$. For $\eta > 0$ we consider the cell problems (1.1) and

$$H_\eta(\chi_{\eta,x}, x) = \overline{H}_\eta \text{ in } \mathbb{R}, \tag{5.1}$$

with H_η as in (1.7). We remark that, since we work on the real line and given the assumptions on H, (5.1) has a strictly sublinear at infinity solution; see Lions and Souganidis [23]. Moreover, the solutions of (1.1) and (5.1) are in $C^{1,1}$ with bounds independent of η. Finally we recall that the solution χ of (1.1) is $\mathbb{Z}-$ periodic.

Following the discussion in the subsection about the one-dimensional problem as well as the proof of Theorem 4.1 for $d = 1$, we rewrite (5.1) as the ode

$$\chi'_\eta(x) = H^{-1}(\zeta_\eta(x) + \overline{H}_\eta, x) \ \text{ in } \ \mathbb{R}, \tag{5.2}$$

together with condition

$$\mathbb{E}\int_Q H^{-1}(\zeta_\eta(x)+\overline{H}_\eta,x)dx=0, \tag{5.3}$$

where $r\to H^{-1}(r,x)$ is the same branch for the inverse of H we used for (2.13).
Let $S_\eta:=\overline{H}_\eta-\overline{H}$ and recall that, in view of Corollary 3.2, $0\le -S_\eta\le C\eta$.
Combining (2.14) with $R=1$ and (5.3), we find

$$\mathbb{E}\int_Q H^{-1}(\zeta_\eta(x)+\overline{H}_\eta,x)=\int_Q H^{-1}(\overline{H},x)dx. \tag{5.4}$$

In what follows, for simplicity we assume that $\mathrm{sppt}(\zeta)\subset Q$. Otherwise we need to account for lower order terms in η^2; we leave the details to the reader.
Since, in view of (1.8) and the assumed Bernoulli law, the left hand side of (5.3) can be evaluated explicitly, we rewrite (5.4) as

$$(1-\eta)\int_Q H^{-1}(S_\eta+\overline{H},x)dx+\eta\int_Q H^{-1}(\zeta(x)+S_\eta+\overline{H},x)dx=\int_Q H^{-1}(\overline{H},x)dx. \tag{5.5}$$

The strict convexity of H and the bound on S_η in Corollary 3.4 yield, as in the proof of Theorem 4.1 for $d=1$,

$$|H^{-1}(\zeta(x)+S_\eta+\overline{H},x)-H^{-1}(\zeta(x)+\overline{H},x)-D_rH^{-1}(\zeta(x)+\overline{H},x)S_\eta|\lesssim(S_\eta)^2\lesssim\eta^2$$

and

$$|H^{-1}(S_\eta+\overline{H},x)-H^{-1}(\overline{H},x)-D_rH^{-1}(\overline{H},x)S_\eta|\lesssim(S_\eta)^2\lesssim\eta^2$$

Integrating over Q and using that

$$\int_Q\left(D_rH^{-1}(\zeta(x)+S_\eta+\overline{H},x)-D_rH^{-1}(\overline{H},x)\right)dx\lesssim 1$$

in (5.5), we get

$$\eta\left[\int_Q(H^{-1}(\zeta(x)+\overline{H},x)-H^{-1}(\overline{H},x))dx\right]+S_\eta\left[\int_Q D_rH^{-1}(\overline{H},x))dx+\mathrm{O}(\eta)\right]=\mathrm{O}(S_\eta^2),$$

and, since $|S_\eta|=\mathrm{O}(\eta)$, as $\eta\to0$,

$$\frac{S_\eta}{\eta}\to-\Big(\int_Q\frac{1}{D_pH(D\chi,x)}dx\Big)^{-1}\int_{\mathrm{sppt}(\zeta)}\left(H^{-1}(\zeta(x)+\overline{H},x)-H^{-1}(\overline{H},x)\right)dx.$$

$\square$

The multidimensional random problem. In higher dimensions, that is when $d\ge2$, the proof is rather delicate and more involved. It is based on constructing a suitable (random) trajectory γ along which it is possible to control the quantity $\int_0^t(L(\dot\gamma,\gamma)+\zeta_\eta(\gamma)+\overline{H})ds$. This is accomplished combining the optimal trajectories of the unperturbed and the "one bump" problems.
The proof is divided into four parts. In the first we introduce some notation, in the second we explain the construction of the random trajectory and in the third we provide the key estimates. The argument is completed in the fourth part.

Notation. In what follows we denote by $\overline{\gamma}^x$ and $\widetilde{\gamma}^x$ the Borel measurable with respect to x optimal paths for χ and χ_∞, that is, for any $x \in \mathbb{R}^d$ and any $0 \le s \le t$,

$$\chi(\overline{\gamma}^x(s)) = \int_s^t \left(L(\dot{\overline{\gamma}}^x(\tau), \overline{\gamma}^x(\tau)) + \overline{H}\right) d\tau + \chi(\overline{\gamma}^x(t))m$$

and

$$\chi_\infty(\widetilde{\gamma}^x(s)) = \int_s^t \Big(L(\dot{\widetilde{\gamma}}^x(\tau), \widetilde{\gamma}^x(\tau)) + \overline{H} + \zeta(\widetilde{\gamma}^x(\tau)\Big) d\tau + \chi_\infty(\widetilde{\gamma}^x(t)).$$

Set

$$\zeta_\infty(z) := \sum_{k\in\mathbb{Z}^d} \zeta(z-k),$$

and note for later use that

$$\mathbb{E}\left[\zeta_\eta(x)\right] = \eta\zeta_\infty(x).$$

Fix $D > 0$ be such that $\mathrm{sppt}(\zeta) \subset B(0, D)$ and $\varepsilon > 0$. Then Lemma 2.2 and Corollary 2.4 imply the existence of $R_0 \ge D$ and $T_0 > 0$ such that

$$\chi_\infty(x) = \chi(x) + c \text{ for any } x \in \mathbb{R}^d \text{ with } \langle x, e\rangle \ge R_0, \tag{5.6}$$

$$(\chi_\infty - \chi)(x) \le c + \varepsilon \ \text{ for any } x \in \mathbb{R}^d \text{ with } \langle x, e\rangle \le -R_0, \tag{5.7}$$

and, if γ is a trajectory such that

$$\int_0^t \left(L(\dot\gamma, \gamma) + \overline{H}\right) ds \le 2(\|\chi\|_\infty + \|\chi_\infty\|_\infty),$$

then, for all $t \ge 0$,

$$(i)\ \inf_{s\ge t}\langle \gamma(s) - \gamma(t), e\rangle \ge -R_0 \quad \text{and} \quad (ii)\ \inf_{s\ge t+T_0}\langle \gamma(s) - \gamma(t), e\rangle \ge R_0. \tag{5.8}$$

We also set

$$R_0' := R_0 + \|D_pH(D\chi, \cdot)\|_\infty T_0, \tag{5.9}$$

and, for $s < t$, E_s^-, E_t^+ and $E_{s,t}$ are the sets

$$\begin{cases} E_s^- := \left\{y \in \mathbb{R}^d : \langle y, e\rangle < s\right\},\ E_t^+ := \left\{y \in \mathbb{R}^d : \langle y, e\rangle \ge t\right\}, \text{ and} \\ E_{s,t} := \left\{y \in \mathbb{R}^d : s \le \langle y, e\rangle < t\right\}. \end{cases}$$

Finally, $\mathcal{F}_s^-$, $\mathcal{F}_t^+$ and $\mathcal{F}_{s,t}$ are the $\sigma-$algebras generated by the random variables X_k with $k \in \mathbb{Z}^d$ and $k \in E_s^-$, $k \in E_t^+$ and $k \in E_{s,t}$ respectively.

In what follows we fix $R \ge 2R_0'$, $n \in \mathbb{N}$ and $\eta > 0$ and we set, for all $n \in \mathbb{N}$, $r_n := nR$.

Throughout the proof C is a generic constant, which may change from line to line, depends on the data and on ε by the choice of R_0, T_0 or R_0', but not on R, η or n. In addition C_R is a constant which may also depend on R.

The random trajectory. We construct by induction a sequence of random points $(x_n)_{n\in\mathbb{N}}$ and times $(\tau_n)_{n\in\mathbb{N}}$ and, then, on each time interval $[\tau_n,\tau_{n+1}]$, a random trajectory γ.

We define τ_0, x_0 and γ on $[0,\tau_0]$ as follows:

$$\tau_0 := \inf\left\{t\geq 0,\ \inf_{s\geq t}\langle \overline{\gamma}^0(s), e\rangle \geq r_0\right\},\ x_0 := \overline{\gamma}^0(\tau_0) \text{ and } \gamma := \overline{\gamma}^0 \text{ on } \ [0,\tau_0].$$

Note that x_0, τ_0 are deterministic with $\gamma(0)=0$, $\langle \gamma(\tau_0), e\rangle = r_0 = 0$.

Assuming next that x_n and τ_n are known, we find x_{n+1}, τ_{n+1} and γ on $[\tau_n,\tau_{n+1}]$. For this we need to consider three disjoint events $A_{n,0}$, $A_{n,1}$ and $A_{n,2}$. Roughly speaking, in the event $A_{n,0}$, the perturbation ζ_η vanishes on the trajectory $\overline{\gamma}^{x_n}$ in the set $E_{r_n+R_0',r_{n+1}-R_0}$. In $A_{n,1}$, the trajectory $\overline{\gamma}^{x_n}$ encounters only one bump in $E_{r_n+R_0',r_{n+1}-R_0}$, and there are no other bump in a large neighborhood of the trajectory. The last event is the complement of the other two.

More precisely, recalling that $r_n = nR$ and that the radius D is such that $\mathrm{sppt}(\zeta)\subset B(0,D)$, the event $A_{n,0}$ is defined by the property that there is no $k\in\mathbb{Z}^d\cap E_{r_n+R_0',r_{n+1}-R_0}$ with $X_k = 1$ and $|k-\overline{\gamma}^{x_n}(t)|\leq D$ for some $t\geq 0$. In the event $A_{n,1}$, there exists a unique $\widehat{k}_n\in\mathbb{Z}^d\cap E_{r_n+R_0',r_{n+1}-R_0}$ with $X_{\widehat{k}_n}=1$ and $|\widehat{k}_n-\overline{\gamma}^{x_n}(t)|\leq D$ for some $t\geq 0$, but there is no other $k'\in\mathbb{Z}^d\cap E_{r_n+R_0',r_{n+1}-R_0}\cap B(x_n,K_R)$ such that $X_{k'}=1$; here $K_R>0$ is a large constant depending on R defined in (5.13) below. Finally, $A_{n,2}=\Omega\backslash(A_{n,0}\cup A_{n,1})$.

In $A_{n,0}\cup A_{n,2}$,

$$\tau_{n+1} := \tau_n + \inf\{t\geq 0 : \inf_{s\geq t}\langle \overline{\gamma}^{x_n}(s), e\rangle \geq r_{n+1}\}, \quad x_{n+1} := \overline{\gamma}^{x_n}(\tau_{n+1}-\tau_n)$$

and

$$\gamma := \overline{\gamma}^{x_n}(\cdot - \tau_n) \text{ in } [\tau_n,\tau_{n+1}].$$

In $A_{n,1}$, we set $\gamma := \overline{\gamma}^{x_n}(\cdot-\tau_n)$ in $[\tau_n,\tau_n+T_0]$, $z_n := \overline{\gamma}^{x_n}(T_0)$,

$$\tau_{n+1} := \tau_n + T_0 + \inf\{t\geq 0 : \inf_{s\geq t}\langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(s)+\widehat{k}_n, e\rangle \geq r_{n+1}\},$$

where $\widehat{k}_n$ is given in the definition of $A_{n,1}$,

$$x_{n+1} := \widetilde{\gamma}^{z_n-\widehat{k}_n}(\tau_{n+1}-\tau_n-T_0)+\widehat{k}_n \text{ and } \gamma := \widetilde{\gamma}^{z_n-\widehat{k}_n}(\cdot-\tau_n-T_0)+\widehat{k}_n \text{ in } [\tau_n+T_0,\tau_{n+1}].$$

Note that, by definition, for all $n\in\mathbb{N}$,

$$\gamma(0)=0,\ \gamma(\tau_n)=x_n \text{ and } \langle x_n, e\rangle = r_n.$$

Moreover, by Corollary 2.3, the times τ_n are finite.

The key properties of the construction. In the next four lemmata we study the properties of the construction above that are needed to complete the proof of the theorem.

Lemma 5.2. *For any $n\in\mathbb{N}$ and all $t>0$, the trajectory $\overline{\gamma}^{x_n}$ remains in $E^+_{r_n}$.*

Proof. We argue using induction. Observe that, in view of the choice of x_0 and τ_0, the claim is immediate for $n=0$, and assume that the result holds for some n.

The choice of τ_{n+1} in $A_{n,0}\cup A_{n,2}$, yields that, for all $t\geq 0$, $\overline{\gamma}^{x_{n+1}}(t)=\overline{\gamma}^{x_n}(t+\tau_{n+1})$ belongs to $E^+_{r_{n+1}}$.

In view of the definition of τ_{n+1} in $A_{n,1}$, to conclude we need to show that, for all $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\widetilde{\gamma}^{z_n - \widehat{k}_n}(t) + \widehat{k}_n = \overline{\gamma}^{x_{n+1}}(t - \tau_{n+1} + \tau_n + T_0). \tag{5.10}$$

Note first that (5.10) holds for $t = \tau_{n+1} - \tau_n - T_0$. Then the definition of τ_{n+1} and the fact that $\widehat{k}_n \in E_{r_n + R_0', r_{n+1} - R_0}$ yield that, for all $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\langle \widetilde{\gamma}^{z_n - \widehat{k}_n}(t), e\rangle \geq r_{n+1} - \langle \widehat{k}_n, e\rangle \geq R_0. \tag{5.11}$$

It follows, in view of (5.6), that for all $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\chi_\infty(\widetilde{\gamma}^{z_n - \widehat{k}_n}(t)) = \chi(\widetilde{\gamma}^{z_n - \widehat{k}_n}(t)). \tag{5.12}$$

The optimality of $\widetilde{\gamma}^{z_n - \widehat{k}_n}$ for χ_∞ implies that, for any $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\chi_\infty(x_{n+1}) = \int_{\tau_{n+1} - \tau_n - T_0}^{t} (L(\dot{\widetilde{\gamma}}^{z_n - \widehat{k}_n}, \widetilde{\gamma}^{z_n - \widehat{k}_n}) + \overline{H} + \zeta(\widetilde{\gamma}^{z_n - \widehat{k}_n}))ds + \chi_\infty(\widetilde{\gamma}^{z_n - \widehat{k}_n}(t)),$$

where, by (5.11), $\zeta(\widetilde{\gamma}^{z_n - \widehat{k}_n}(s)) = 0$ on $[\tau_{n+1} - \tau_n - T_0, t]$.
Hence, using (5.12), we find

$$\chi(x_{n+1}) = \int_{\tau_{n+1} - \tau_n - T_0}^{t} \left(L(\dot{\widetilde{\gamma}}(s), \widetilde{\gamma}^{z_n - \widehat{k}_n}(s)) + \overline{H}\right) ds + \chi(\widetilde{\gamma}(t)).$$

It follows that $\widetilde{\gamma}^{z_n - \widehat{k}_n}$ is optimal for χ in $[\tau_{n+1} - \tau_n - T_0, +\infty)$, and, in view of the uniqueness of the optimal solution, we obtain that, for all $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\widetilde{\gamma}^{z_n - \widehat{k}_n}(t) = \overline{\gamma}^{\widetilde{\gamma}^{z_n - \widehat{k}_n}(\tau_{n+1} - \tau_n - T_0)}(t - \tau_{n+1} + \tau_n + T_0).$$

Using that the map $x \to \overline{\gamma}^x$ is periodic, we get that, for any $t \geq \tau_{n+1} - \tau_n - T_0$,

$$\widetilde{\gamma}^{z_n - \widehat{k}_n}(t) + \widehat{k}_n = \overline{\gamma}^{\widetilde{\gamma}^{z_n - \widehat{k}_n}(\tau_{n+1} - \tau_n - T_0) + \widehat{k}_n}(t - \tau_{n+1} + \tau_n + T_0) = \overline{\gamma}^{x_{n+1}}(t - \tau_{n+1} + \tau_n + T_0),$$

which is (5.10). □

Lemma 5.3. *For any $n \in \mathbb{N}_0$, x_n, τ_n and the restriction of γ to $[0, \tau_n]$ are $\mathcal{F}^-_{r_n - R_0}$-measurable, while the events $A_{n,0}$, $A_{n,1}$ and $A_{n,2}$ are $\mathcal{F}^-_{r_{n+1} - R_0}$-measurable.*

Proof. We argue again by induction. The claim is true for $n = 0$ since x_0, τ_0 and the restriction of γ to $[0, \tau_0]$ are deterministic.
We assume next that x_n, τ_n and the restriction of γ to $[0, \tau_n]$ are $\mathcal{F}^-_{r_n - R_0}$-measurable. Knowing x_n and τ_n, it follows that $A_{n,0}$, $A_{n,1}$ and $A_{n,2}$ belong to $\mathcal{F}^-_{r_{n+1} - R_0}$, and the induction assumption, implies that $A_{n,0}$, $A_{n,1}$ and $A_{n,2}$ are $\mathcal{F}^-_{r_{n+1} - R_0}$-measurable. It follows from their definition that x_{n+1}, τ_{n+1} and the restriction of γ to $[0, \tau_{n+1}]$ are $\mathcal{F}^-_{r_{n+1} - R_0}$-measurable. □

Lemma 5.4. *For any $n \in \mathbb{N}_0$, $\gamma(t)$ belongs to $E^+_{r_n}$ for all $t \geq \tau_n$.*

Proof. It is enough to show that, for any $n \in \mathbb{N}_0$, $\gamma(t)$ belongs to E^+_n for $t \in [\tau_n, \tau_{n+1}]$. Fix $n \in \mathbb{N}$. Lemma 5.2 implies that, for all $t \geq 0$, $\overline{\gamma}^{x_n}(t) \in E^+_{r_n}$, and the claim is clear in $A_{n,0} \cup A_{n,2}$. In $A_{n,1}$, we have $\gamma(t) = \overline{\gamma}^{x_n}(t - \tau_n)$ for $t \in [\tau_n, \tau_n + T_0]$, and again $\gamma(t) \in E^+_{r_n}$ in this interval. Moreover, (5.8)(ii) yields that

$$\langle z_n, e\rangle = \langle \overline{\gamma}^{x_n}(T_0), e\rangle \geq \langle x_n, e\rangle + R_0 = r_n + R_0,$$

and, hence, the choice of R_0 in (5.8) yields

$$\inf_{s\geq 0}\langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(s)+\widehat{k}_n, e\rangle \geq \langle z_n, e\rangle - R_0 = r_n.$$

□

Lemma 5.5. *There exists $C_0>0$, which is independent of ε, R, n and η, such that, for all $n\in\mathbb{N}$, $C_0^{-1}R \leq \tau_{n+1}-\tau_n \leq C_0 R$. In particular, as $n\to\infty$, and almost surely, $\tau_n\to+\infty$.*

Proof. Since $\langle x_n, e\rangle = \langle \gamma(\tau_n), e\rangle = nR$, we deduce that

$$\begin{aligned} R = r_{n+1}-r_n &= \langle \gamma(\tau_{n+1})-\gamma(\tau_n), e\rangle \leq (\tau_{n+1}-\tau_n)\|\dot\gamma\|_\infty \\ &\leq (\tau_{n+1}-\tau_n)(\|D_pH(\cdot, D\chi)\|_\infty + \|D_pH(\cdot, D\chi_\infty)\|_\infty), \end{aligned}$$

which proves that $(\tau_{n+1}-\tau_n)\geq C_0^{-1}R$ for some $C_0>0$.

Recall that, for all $x\in\mathbb{R}^d$ and $t\geq 0$,

$$\int_0^t \left(L(\dot{\overline{\gamma}}^x, \overline{\gamma}^x)+\overline{H}\right)ds \leq 2\|\chi\|_\infty.$$

Then Lemma 2.2 yields $\overline{T}_0>0$ such that, for all $t\geq 0$,

$$\inf_{s\geq t+\overline{T}_0}\langle \overline{\gamma}(s)-\overline{\gamma}(t), e\rangle \geq 1.$$

Similarly, since, for all $t\geq 0$,

$$\int_0^t \left(L(\dot{\widetilde{\gamma}}^x, \widetilde{\gamma}^x)+\overline{H}\right)ds \leq 2\|\chi_\infty\|_\infty,$$

it follows that

$$\inf_{t\geq 0}\inf_{s\geq t+\overline{T}_0}\langle \widetilde{\gamma}(s)-\widetilde{\gamma}(t), e\rangle \geq 1.$$

Hence, in the event $A_{n,0}\cup A_{n,2}$, where $\gamma(t)=\overline{\gamma}^{x_n}(t-\tau_n)$ for $t\in[\tau_n,\tau_{n+1}]$, we have

$$R=\langle \gamma(\tau_{n+1})-\gamma(\tau_n), e\rangle \geq [(\tau_{n+1}-\tau_n)/\overline{T}_0],$$

where $[a]$ denotes the integer part of a.

In the event $A_{n,1}$, $\gamma=\overline{\gamma}^x(\cdot-\tau_n)$ on $[\tau_n,\tau_n+T_0]$ and $\gamma=\widetilde{\gamma}^{z_n}(\cdot-\tau_n-T_0)$ on $[\tau_n+T_0,\tau_{n+1}]$, where $z_n=\overline{\gamma}(\tau_n+T_0)$. Thus

$$\langle \gamma(\tau_{n+1})-\gamma(\tau_n+T_0), e\rangle \geq [(\tau_{n+1}-\tau_n-T_0)/\overline{T}_0],$$

while

$$\langle \gamma(\tau_n+T_0)-\gamma(\tau_n), e\rangle \geq [T_0/\overline{T}_0].$$

Since, from the first part of the proof, $\tau_{n+1}-\tau_n$ is large for large R, combining the inequalities above, we find that, for a suitable choice of C_0,

$$R\geq C_0^{-1}(\tau_{n+1}-\tau_n).$$

□

We now define the constant K_R that was used in the construction of the random trajectory as

$$K_R := C_0R\left(\|D_pH(\cdot, D\chi)\|_\infty + \|D_pH(\cdot, D\chi_\infty)\|_\infty\right), \tag{5.13}$$

and remark, for later use, that, in $[\tau_n,\tau_{n+1}]$, $|\gamma(t)-x_n|\leq K_R$.

We also emphasize that the construction of C_0 in the proof of Lemma 5.5 is deterministic. indeed, it does not depend on the definition of the random sets $A_{n,0}$, $A_{n,1}$ and $A_{n,2}$, but only on the possible expressions the trajectory γ can take in these events. In particular, the definition of K_R is not circular.

Lemma 5.6. *There exists $C_1 > 0$, which is independent of ε, R, n and η, such that*

$$\mathbb{P}\left[A_{n,1}\right] \leq C_1 R\eta.$$

The intuition behind the lemma is quite clear. The set $A_{n,1}$ is contained in the event that there is at least one bump $\zeta(\cdot - k)$, which is both near the trajectory γ and belongs to $E_{r_n,r_{n+1}}$. Since $|k| \lesssim R$, $\mathbb{P}[A_{1,n}] \lesssim R\eta$.

The proof of Lemma 5.6. Let S be the random set

$$S := \left\{ k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}} : \inf_{t\geq 0} |\overline{\gamma}^{x_n}(t) - k| \leq D \right\}.$$

Lemma 2.2 implies the existence of a constant $\overline{T}_0$, independent of ε, such that

$$\inf_{s\geq t+\overline{T}_0} \langle \overline{\gamma}^x(s) - \overline{\gamma}^x(t), e\rangle \geq 1.$$

Next we discretize $\overline{\gamma}^x$ with step size $t_l = l\overline{T}_0$, for some $l \in \mathbb{N}$, such that

$$|\overline{\gamma}^{x_n}(t) - \overline{\gamma}^{x_n}(l\overline{T}_0)| \leq M\overline{T}_0,$$

where $M := \|D_p H(D\chi, \cdot)\|_\infty$.
It follows that

$$S \subset \left\{ k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}} : \inf_{l\in\mathbb{N}} |\overline{\gamma}^{x_n}(l\overline{T}_0) - k| \leq D + M\overline{T}_0 \right\}.$$

Note also that since, for any $l \geq L_R := [R + D + M\overline{T}_0] + 2$, we have

$$\langle \overline{\gamma}^{x_n}(l\overline{T}_0), e\rangle > \langle x_n, e\rangle + R + D + M\overline{T}_0 + 1 \geq r_{n+1} + D + M\overline{T}_0 + 1,$$

if $k \in \mathbb{Z}^d$ is such that $|\overline{\gamma}^{x_n}(l\overline{T}_0) - k| \leq D + M\overline{T}_0$, then $k \in E^+_{r_n+1}$.
Hence

$$S \subset \left\{ k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}} : \inf_{l=0,\dots,L_R} |\overline{\gamma}^{x_n}(t_l) - k| \leq D + M\overline{T}_0 \right\}.$$

Since

$$A_{n,1} \subset \{\exists k \in S,\ X_k = 1\},$$

we deduce that

$$\begin{aligned} \mathbb{P}\left[A_{n,1}\right] &\leq \mathbb{P}\left[\exists k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}},\ \exists l = \{0,\dots,L_R\},\ |\overline{\gamma}^{x_n}(t_l) - k| \leq D + M\overline{T}_0,\ X_k = 1\right] \\ &\leq \sum_{l=0}^{L_R} \mathbb{P}\left[\exists k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}},\ |\overline{\gamma}^{x_n}(t_l) - k| \leq D + M\overline{T}_0,\ X_k = 1\right]. \end{aligned}$$

The $\mathcal{F}^-_{r_n-R_0}$-measurability of x_n implies that the event $\{X_k = 1\}$ with $k \in \mathbb{Z}^d \cap E_{r_n,r_{n+1}}$ is independent of x_n, and, thus

$$\mathbb{P}\left[A_{n,1}\right] \leq L_R C_d (D + M\overline{T}_0)^d \eta,$$

where C_d depends only on the dimension.
Then, since $L_R = [R + D + M\overline{T}_0] + 2$, we may conclude. □

Proof of Theorem 5.1. It is known (see [28]) that, if

$$\theta(t) := \inf_{\xi(0)=0} \int_0^t \Big(L(\dot\xi(s), \xi(s)) + \overline{H} + \zeta_\eta(\xi(s)) \Big)\, ds,$$

then, almost surely,

$$\lim_{t\to+\infty} \frac{\theta(t)}{t} = \overline{H} - \overline{H}_\eta.$$

In particular, in view of Lemma 5.5, we have

$$\lim_{n\to+\infty} \mathbb{E}\left[\frac{\theta(\tau_n)}{\tau_n}\right] = \overline{H} - \overline{H}_\eta.$$

If γ is the trajectory built in a previous subsection, then

$$\overline{H} - \overline{H}_\eta \le \limsup_{n\to+\infty} \mathbb{E}\left[\frac{1}{\tau_n}\int_0^{\tau_n} \big(L(\dot\gamma,\gamma) + \overline{H} + \zeta_\eta(\gamma)\big)\, ds\right]. \tag{5.14}$$

To estimate the right-hand side the inequality above, which is the core of the proof, we need to establish three more auxiliary results, which we formulate next as separate lemmata.
We set

$$I_n := \int_{\tau_n}^{\tau_{n+1}} \big(L(\dot\gamma,\gamma) + \overline{H} + \zeta_\eta(\gamma)\big)\, ds,$$

and we successively estimate I_n in $A_{n,0}$, $A_{n,2}$ and $A_{n,1}$ noting that the estimate in the last set is the hardest to establish.

Lemma 5.7. *There exists a nonnegative random variable $R_{n,0}$ such that*

$$\mathbf{1}_{A_{n,0}} I_n \;=\; \mathbf{1}_{A_{n,0}}(\chi(x_n) - \chi(x_{n+1})) + R_{n,0} \quad \text{with} \quad \mathbb{E}\left[R_{n,0}\right] \le C\eta.$$

Proof. Recall that $\gamma = \overline{\gamma}^{x_n}(\cdot - \tau_n)$ in $A_{n,0}$. It follows that

$$\begin{aligned}
\mathbf{1}_{A_{n,0}} I_n \;&=\; \mathbf{1}_{A_{n,0}} \left(\int_0^{\tau_{n+1}-\tau_n} \big(L(\dot{\overline{\gamma}}^{x_n}(s), \overline{\gamma}^{x_n}(s)) + \overline{H} + \zeta_\eta(\overline{\gamma}^{x_n}(s))\big)\, ds \right) \\
&=\; \mathbf{1}_{A_{n,0}} \left(\chi(x_n) - \chi(x_{n+1}) + \int_0^{\tau_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(s)) ds \right).
\end{aligned}$$

Let

$$R_{n,0} := \mathbf{1}_{A_{n,0}} \int_0^{\tau_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(s)) ds.$$

Since $A_{n,0}$ is the event that there is no $k \in \mathbb{Z}^d \cap E_{r_n+R_0', r_{n+1}-R_0}$ with $X_k = 1$ and $|k - \overline{\gamma}^{x_n}(t)| \le D$ for some $t \ge 0$,

$$\zeta_\eta(\gamma(t)) = 0 \text{ if } \gamma(t) = \overline{\gamma}^{x_n}(t-\tau_n) \in E_{r_n+R_0'+D, r_{n+1}-R_0-D} \text{ and } t \in [\tau_n, \tau_{n+1}].$$

Let

$$\overline{\tau}_{n+1} := \tau_n + \inf\{t \ge 0,\ \inf_{s\ge t} \langle \overline{\gamma}^{x_n}(s), e\rangle \ge r_{n+1}\},$$

and recall that, in view of Lemma 5.3, x_n and τ_n are $\mathcal{F}^-_{r_n-R_0}$ measurable. Then $\overline{\tau}_{n+1}$ is also $\mathcal{F}^-_{r_n-R_0}$-measurable and equals τ_{n+1} in $A_{n,0}$.
We now estimate for how long the trajectory $\overline{\gamma}^{x_n}$ remains in $E_{r_n+R_0'+D, r_{n+1}-R_0-D}$. Using (5.8)(ii) we find that, if $C := (1 + [(R_0' + D)/R_0])$, then, for all $t \ge CT_0$,

$$\langle \overline{\gamma}^{x_n}(t) - x_n, e\rangle \ge R_0' + D,$$

and, similarly, if $t \le \overline{\tau}_{n+1} - CT_0$, then

$$\langle x_{n+1} - \overline{\gamma}^{x_n}(t), e\rangle \ge R_0 + D.$$

It follows that $\overline{\gamma}^{x_n}(t) \in E_{r_n+R_0'+D, r_n-R_0-D}$ for $t \in [CT_0, \overline{\tau}_{n+1} - CT_0]$, and, hence,

$$\mathbb{E}\left[\mathbf{1}_{A_{n,0}} \int_{\tau_n}^{\tau_{n+1}} \zeta_\eta(\gamma(t))dt\right] \le \mathbb{E}\left[\int_0^{CT_0} \zeta_\eta(\overline{\gamma}^{x_n}(t))dt + \int_{\overline{\tau}_{n+1}-\tau_n-CT_0}^{\overline{\tau}_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(t))dt\right].$$

Using that $\overline{\gamma}^{x_n}$ remains in $E^+_{r_n}$ for positive times, x_n is $\mathcal{F}^-_{r_n-R_0}$-measurable and the map ζ_η is independent of $\mathcal{F}^-_{r_n-R_0}$ in $E^+_{r_n}$ with $\mathbb{E}[\zeta_\eta(z)] = \eta\zeta_\infty(z)$, we find

$$\begin{aligned}\mathbb{E}\left[\int_0^{CT_0} \zeta_\eta(\overline{\gamma}^{x_n}(t))dt\right] &= \mathbb{E}\left[\int_0^{CT_0} \mathbb{E}\left[\zeta_\eta(\overline{\gamma}^{x}(t)) \mid \mathcal{F}^-_{r_n-R_0}\right]_{x=x_n} dt\right] \\ &= \mathbb{E}\left[\eta \int_0^{CT_0} \zeta_\infty(\overline{\gamma}^{x_n}(t))dt\right] \le C\eta\|\zeta_\infty\|_\infty T_0.\end{aligned}$$

In the same way, since x_n, τ_n and $\overline{\tau}_{n+1}$ are $\mathcal{F}^-_{r_n-R_0}$ measurable,

$$\mathbb{E}\left[\int_{\overline{\tau}_{n+1}-\tau_n-CT_0}^{\overline{\tau}_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(t))dt\right] \le C\eta\|\zeta_\infty\|_\infty T_0.$$

The claim follows. □

Next we estimate I_n in $A_{n,2}$.

Lemma 5.8. *There exists a nonnegative random variable $R_{n,2}$ such that*

$$\mathbf{1}_{A_{n,2}} I_n \;=\; \mathbf{1}_{A_{n,2}}\left(\chi(x_n) - \chi(x_{n+1})\right) + R_{n,2} \quad \text{and} \quad \mathbb{E}\left[R_{n,2}\right] \le C_R\eta^2.$$

Proof. It is immediate that

$$\mathbf{1}_{A_{n,2}} I_n \;=\; \mathbf{1}_{A_{n,0}}\left(\chi(\gamma(x_n) - \chi(x_{n+1})\right) + R_{n,2},$$

with

$$R_{n,2} := \mathbf{1}_{A_{n,2}} \int_0^{\tau_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(s))ds.$$

In the event $A_{n,2}$, there are at least two different $k, k' \in \mathbb{Z}^d \cap E_{r_n+R_0', r_{n+1}-R_0} \cap B(x_n, K_R)$ with $X_k = X_{k'} = 1$. Since x_n is independent of $\mathcal{F}_{r_n}$,

$$\mathbb{P}\left[A_{n,2}\right] \le C_R\eta^2.$$

It follows from Lemma 5.5 that $\tau_{n+1} - \tau_n \le C_0 R$, and, hence,

$$\begin{aligned}\mathbb{E}\left[R_{n,2}\right] &= \mathbb{E}\left[\mathbf{1}_{A_{n,2}} \int_0^{\tau_{n+1}-\tau_n} \zeta_\eta(\overline{\gamma}^{x_n}(t))dt\right] \\ &\le \mathbb{E}\left[\mathbf{1}_{A_{n,2}} \int_0^{\tau_{n+1}-\tau_n} \zeta_\infty(\overline{\gamma}^{x_n}(t))dt\right] \le C_0 R\|\zeta_\infty\|_\infty \mathbb{P}\left[A_{n,2}\right],\end{aligned}$$

and the claim follows. □

The next lemma is about $A_{n,1}$.

Lemma 5.9. *Let C_1 be as in Lemma 5.6. There exists a nonnegative random variable $R_{n,1}$ such that*

$$\mathbf{1}_{A_{n,1}} I_n \le \mathbf{1}_{A_{n,1}}\left(\chi(x_n) - \chi(x_{n+1})\right) + R_{n,1} \quad \textit{and} \quad \mathbb{E}\left[R_{n,1}\right] \le (C + C_1 R\varepsilon)\eta + C_R\eta^2.$$

Proof. In the event $A_{n,1}$, we have set $\gamma = \overline{\gamma}^{x_n}(\cdot - \tau_n)$ on $[\tau_n, \tau_n + T_0]$ and $\gamma = \widetilde{\gamma}^{z_n - \widehat{k}_n}(\cdot - \tau_n - T_0) + \widehat{k}_n$ on $[\tau_n + T_0, \tau_{n+1}]$, where $z_n := \overline{\gamma}^{x_n}(T_0)$. So we can write

$$\mathbf{1}_{A_{n,1}} I_n = A + B$$

with

$$A := \mathbf{1}_{A_{n,1}} \int_0^{T_0} \left(L(\dot{\overline{\gamma}}^{x_n}(s), \overline{\gamma}^{x_n}(s)) + \overline{H} + \zeta_\eta(\overline{\gamma}^{x_n}(s))\right) ds,$$

and

$$B := \mathbf{1}_{A_{n,1}} \int_0^{\tau_{n+1}-\tau_n-T_0} \left(L(\dot{\widetilde{\gamma}}^{z_n-\widehat{k}_n}(s), \widetilde{\gamma}^{z_n-\widehat{k}_n}(s) + \widehat{k}_n) + \overline{H} + \zeta_\eta(\widetilde{\gamma}^{z_n-\widehat{k}_n}(s) + \widehat{k}_n)\right) ds.$$

To estimate A we argue as in Lemma 5.7. Indeed,

$$A = \mathbf{1}_{A_{n,1}} \left(\chi(x_n) - \chi(z_n)\right) + R'_{n,1}$$

with

$$R'_{n,1} := \mathbf{1}_{A_{n,1}} \int_0^{T_0} \zeta_\eta(\overline{\gamma}^{x_n}(s)) ds.$$

It then follows, as in the proof of Lemma 5.7, that

$$\mathbb{E}\left[R'_{n,1}\right] \leq C\eta.$$

We now turn to the estimate for B. In the event $A_{n,1}$, there exists a unique $\widehat{k}_n \in \mathbb{Z}^d \cap E_{r_n+R'_0, r_{n+1}-R_0}$ with $X_{\widehat{k}_n} = 1$ and $|\widehat{k}_n - \overline{\gamma}^{x_n}(t)| \leq D$ for some $t \geq 0$, and there is no other $k' \in \mathbb{Z}^d \cap E_{r_n+R'_0, r_{n+1}-R_0} \cap B(x_n, K_R)$ such that $X_{k'} = 1$. Therefore

$$\zeta_\eta(x) = \zeta(x - \widehat{k}_n) \text{ if } x \in E_{r_n+R'_0+D, r_{n+1}-R_0-D} \text{ and } |x - x_n| \leq K_R. \tag{5.15}$$

The definition of $\widetilde{\gamma}^{z_n-\widehat{k}_n}$ yields

$$\begin{aligned}
&\int_0^{\tau_{n+1}-\tau_n-T_0} \left(L(\dot{\widetilde{\gamma}}^{z_n-\widehat{k}_n}(s), \widetilde{\gamma}^{z_n-\widehat{k}_n}(s)) + \overline{H} + \zeta(\widetilde{\gamma}^{z_n-\widehat{k}_n}(s))\right) ds \\
&\qquad = \chi_\infty(z_n - \widehat{k}_n) - \chi_\infty(\widetilde{\gamma}^{z_n-\widehat{k}_n}(\tau_{n+1} - \tau_n - T_0)) = \chi_\infty(z_n - \widehat{k}_n) - \chi_\infty(\gamma(\tau_{n+1} - \widehat{k})).
\end{aligned}$$

Moreover, since $\widehat{k}_n \in E_{r_n+R'_0, r_{n+1}-R_0}$ while $|z_n - x_n| \leq \|D_pH(\cdot, D\chi)\|_\infty T_0$, using the R'_0 in (5.9), we have

$$\langle z_n - \widehat{k}_n, e\rangle \leq \langle x_n - \widehat{k}_n, e\rangle + \|D_pH(\cdot, D\chi)\|_\infty T_0 \leq -R'_0 + \|D_pH(\cdot, D\chi)\|_\infty T_0 \leq -R_0.$$

It then follows from (5.7) and the periodicity of χ that

$$\chi_\infty(z_n - \widehat{k}_n) \leq \chi(z_n - \widehat{k}_n) + c + \varepsilon = \chi(z_n) + c + \varepsilon.$$

The definition of τ_{n+1} implies that $\langle \gamma(\tau_{n+1}), e\rangle = r_{n+1}$ and, hence,

$$\langle \gamma(\tau_{n+1}) - \widehat{k}_n, e\rangle \geq r_{n+1} - (r_{n+1} - R_0) = R_0,$$

and, in view of (5.6) and the periodicity of χ,

$$\chi_\infty(\gamma(\tau_{n+1}) - \widehat{k}_n) = \chi(\gamma(\tau_{n+1}) - \widehat{k}_n) + c = \chi(x_{n+1}) + c.$$

Collecting the above inequalities and using the periodicity in space of L, we find that, in $A_{n,1}$,

$$\begin{array}{l}\displaystyle\int_{\tau_n+T_0}^{\tau_{n+1}} \Big(L(\dot\gamma(s),\gamma(s))+\zeta(\gamma(s)-\widehat{k}_n)\Big)\,ds\\ \displaystyle = \int_0^{\tau_{n+1}-\tau_n-T_0} \Big(L(\dot{\widetilde{\gamma}}^{z_n-\widehat{k}_n}(s),\widetilde{\gamma}^{z_n-\widehat{k}_n}(s))+\overline{H}+\zeta(\widetilde{\gamma}^{z_n-\widehat{k}_n}(s))\Big)\,ds\\ \quad \le \chi(z_n)-\chi(x_{n+1})+\varepsilon,\end{array} \tag{5.16}$$

and, hence,

$$B \le \mathbf{1}_{A_{n,1}}\left(\chi(z_n)-\chi(x_{n+1})+\varepsilon\right)+R''_{n,1},$$

where

$$R''_{n,1} := \mathbf{1}_{A_{n,1}} \int_{\tau_n+T_0}^{\tau_{n+1}} \Big(\zeta_\eta(\gamma(s))-\zeta(\gamma(s)-\widehat{k}_n)\Big)\,ds.$$

Recalling (5.15) as well as the fact that $\gamma(t)\in B(x_n,K_R)$ for $t\in[\tau_n,\tau_{n+1}]$, we find

$$R''_{n,1} = \mathbf{1}_{A_{n,1}} \int_{\tau_n+T_0}^{\tau_{n+1}} \mathbf{1}_{\gamma\notin E_{r_n+R'_0+D,r_{n+1}-R_0-D}}\zeta_\eta(\gamma(t))dt.$$

Let $A'_{n,1}\subset A_{n,1}$ be the event that there is at least one bump different from $\widehat{k}_n$, in $(E_{r_n,r_n+R'_0}\cup E_{r_{n+1}-R_0,r_{n+1}+R_0})\cap B(x_n,K_R)$. Since $\gamma|_{[\tau_n,\tau_{n+1}]}$ belongs to $E_{r_n,r_{n+1}+R_0}\cap B(x_n,K_R)$, we get

$$\mathbb{E}\left[R''_{n,1}\right] = \mathbb{E}\left[\mathbf{1}_{A'_{n,1}} \int_{\tau_n+T_0}^{\tau_{n+1}} \mathbf{1}_{\gamma\notin E_{r_n+R'_0+D,r_{n+1}-R_0-D}}\zeta_\eta(\gamma(t))dt\right].$$

It follows from (5.8)(ii) that, if $t\ge CT_0$ with $C:=([(R'_0+D)/R_0]+1)$, then

$$\langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(t)+\widehat{k}_n,e\rangle \ge \langle z_n,e\rangle+([(R'_0+D)/R_0]+1)R_0 \ge r_n+R'_0+D.$$

Moreover, since $\langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(\tau_{n+1}-\tau_n-T_0)+\widehat{k}_n,e\rangle = r_{n+1}$, we also find, for $t\le \tau_{n+1}-CT_0$,

$$\begin{array}{rcl}\langle\gamma(t),e\rangle &=& \langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(t-\tau_n-T_0)+\widehat{k}_n,e\rangle\\ &\le& \langle \widetilde{\gamma}^{z_n-\widehat{k}_n}(\tau_{n+1}-\tau_n-T_0)+\widehat{k}_n,e\rangle - R_0-D = r_{n+1}-R_0-D,\end{array}$$

and, thus $\gamma(t)\in E_{r_n+R'_0+D,r_{n+1}-R_0-D}$ for $t\in[\tau_n+CT_0,\tau_{n+1}-CT_0]$.
It follows that

$$\begin{array}{rcl}\mathbb{E}\left[R''_{n,1}\right] &\le& \mathbb{E}\left[\mathbf{1}_{A'_{n,1}}\displaystyle\int_{\tau_n+T_0}^{\tau_n+CT_0}\zeta_\eta(\gamma(t))dt\right]+\mathbb{E}\left[\mathbf{1}_{A'_{n,1}}\displaystyle\int_{\tau_{n+1}-CT_0}^{\tau_{n+1}}\mathbf{1}_{\gamma\notin E_{r_n+R'_0,r_{n+1}-R_0}}\zeta_\eta(\gamma(t))dt\right]\\ &\le& \|\zeta_\eta\|_\infty CT_0\mathbb{P}\left[A'_{n,1}\right].\end{array}$$

In view of the fact that in $A'_{n,1}$ there exist at least two distinct bumps in the set $E_{r_n,r_{n+1}+R_0}\cap B(x_n,K_R)$, we have $\mathbb{E}\left[R''_{n,1}\right]\le C_R\eta^2$.
Writing

$$R_{n,1} = R'_{n,1}+R''_{n,1}+\varepsilon\mathbf{1}_{A_{n,1}},$$

we obtain

$$\mathbf{1}_{A_{n,1}}I_n \le \mathbf{1}_{A_{n,1}}\left(\chi(x_n)-\chi(x_{n+1})\right)+R_{n,1},$$

and, in view of Lemma 5.6 and the above estimates,

$$\mathbb{E}\left[R_{n,1}\right]\le C\eta+C_R\eta^2+\varepsilon\mathbb{P}\left[A_{n,1}\right]\le (C+C_1R\varepsilon)\eta+C_R\eta^2.$$

□

We complete now the proof.

Proof of Theorem 5.1(continued). Combining Lemma 5.7, Lemma 5.8 and Lemma 5.9, we find

$$I_n \le \chi(x_n) - \chi(x_{n+1}) + R_n$$

where $R_n := R_{n,0} + R_{n,1} + R_{n,2}$, and, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[R_n\right] \le (C + C_1 R\varepsilon)\eta + C_R\eta^2.$$

Therefore

$$\int_{\tau_0}^{\tau_n} \left(L(\dot\gamma(t),\gamma(t)) + \overline{H} + \zeta_\eta(\gamma(t))\right) dt \;=\; \sum_{k=0}^{n-1} I_k \le \chi(x_0) - \chi(x_n) + \sum_{k=0}^{n-1} R_k.$$

It follows from Lemma 5.5 that

$$C_0^{-1} Rn \le \tau_n \le C_0 Rn,$$

and, thus,

$$\begin{aligned}\limsup \mathbb{E}\left[\frac{1}{\tau_n}\int_{\tau_0}^{\tau_n} \left(L(\dot\gamma(t),\gamma(t)) + \overline{H} + \zeta_\eta(\gamma(t))\right) ds\right]&\\ \le \limsup_n \frac{C_0}{Rn}\left(2\|\chi\|_\infty + n(C + C_1R\varepsilon)\eta + C_R\eta^2 n\right)&\\ \le\; C_0(CR^{-1} + C_1\varepsilon)\eta + \widetilde{C}_R\eta^2.&\end{aligned}$$

Then, in view of (5.14), we get

$$\overline{H} - \overline{H}_\eta \le C_0(CR^{-1} + C_1\varepsilon)\eta + C_R\eta^2,$$

and, thus,

$$\limsup_{\eta\to0^+} \frac{\overline{H} - \overline{H}_\eta}{\eta} \le C_0(CR^{-1} + C_1\varepsilon).$$

Since C_0 and C_1 are independent on ε and C_0, C_1 and C are independent on R, letting first $R \to +\infty$ and then $\varepsilon \to 0$, we conclude that

$$\limsup_{\eta\to0^+} \frac{\overline{H} - \overline{H}_\eta}{\eta} \le 0.$$

The inequality $\overline{H} - \overline{H}_\eta \ge 0$ then completes the proof. □

References

[1] Y. Achdou and N. Tchou. Hamilton-Jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction. *Communications in Partial Differential Equations*, 40(4): 652–693, 2015.

[2] A. Anantharaman and C. Le Bris. A numerical approach related to defect-type theories for some weakly random problems in homogenization. *Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal*, 9(2): 513–544, 2011.

[3] A. Anantharaman and C. Le Bris. Elements of mathematical foundations for numerical approaches for weakly random homogenization problems. *Communications in Computational Physics,* 11(4): 1103–1143, 2012.

[4] S. N. Armstrong and P. Cardaliaguet. Quantitative stochastic homogenization of viscous Hamilton-Jacobi equations. *Comm. Partial Differential Equations*, 40(3):540–600, 2015.

[5] S. N. Armstrong, P. Cardaliaguet, and P. E. Souganidis. Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations. *J. Amer. Math. Soc.*, 27(2):479–540, 2014.

[6] S. N. Armstrong and P. E. Souganidis. Stochastic homogenization of Hamilton-Jacobi and degenerate Bellman equations in unbounded environments. *J. Math. Pures Appl. (9)*, 97(5):460–504, 2012.

[7] S. N. Armstrong and P. E. Souganidis. Stochastic homogenization of level-set convex Hamilton-Jacobi equations. *Int. Math. Res. Not. IMRN*, 2013(15):3420–3449, 2013.

[8] P. Cardaliaguet, and P. E. Souganidis. Homogenization and enhancement of the G-equation in random environments.*Comm. Pure Appl. Math.,* 60(10):1582–1628, 2013.

[9] P. Cardaliaguet, and P. E. Souganidis. Periodic approximations of the ergodic constants in the stochastic homogenization. *Ann. Inst. H. Poincaré Anal. Non Linéaire,* 32(3): 571Ð-591, 2015.

[10] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[11] M. Duerinckx and A. Gloria. Analyticity of homogenized coefficients under Bernoulli perturbations and the Clausius-Mossotti formulas. *Arch. Ration. Mech. Anal.*, 220(1), 297–361, 2016.

[12] L. C. Evans. The perturbed test function method for viscosity solutions of nonlinear pde. *Proc. Roy. Soc. Edinburgh Sect. A*, 111: 359–375, 1989.

[13] L. C. Evans. Periodic homogenization of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(3-4):245–265, 1992.

[14] A, Fathi. The Weak KAM Theorem in Lagrangian Dynamics. Series: Cambridge Studies in Advanced Mathematics (No. 88).

[15] A. Fathi. Weak KAM from a PDE point of view: viscosity solutions of the Hamilton-Jacobi equation and Aubry set. *Proc. Roy. Soc. Edinburgh Sect. A*,, 142(06), 1193-1236, 2012.

[16] H. Ishii, Almost periodic homogenization of Hamilton-Jacobi equations. International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 600–605, World Sci. Publ., River Edge, NJ, 2000.

[17] E. Kosygina, F. Rezakhanlou, and S. R. S. Varadhan. Stochastic homogenization of Hamilton-Jacobi-Bellman equations. *Comm. Pure Appl. Math.*, 59(10):1489–1521, 2006.

[18] P.-L. Lions. College de France lectures, 2014.

[19] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi equations. *Unpublished preprint*, 1987.

[20] P.-L. Lions and P. E. Souganidis. Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(5):667–677, 2005.

[21] P.-L. Lions and P. E. Souganidis. Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media. *Comm. Partial Differential Equations*, 30(1-3):335–375, 2005.

[22] P.-L. Lions and P. E. Souganidis. Stochastic homogenization of Hamilton-Jacobi and "viscous"-Hamilton-Jacobi equations with convex nonlinearities—revisited. *Commun. Math. Sci.*, 8(2):627–637, 2010.

[23] P.-L. Lions and P. E. Souganidis. Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Communications on pure and applied mathematics*, 56(10):1501–1524, 2003.

[24] P.-L. Lions and P. E. Souganidis. Non periodic perturbations to homogenization. In preparation.

[25] A. Majda and P. E. Souganidis. Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales. *Nonlinearity,* 7(1), 1994.

[26] F. Rezakhanlou and J. E. Tarver. Homogenization for stochastic Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.*, 151(4):277–309, 2000.

[27] R. W. Schwab. Stochastic homogenization of Hamilton-Jacobi equations in stationary ergodic spatio-temporal media. *Indiana Univ. Math. J.*, 58(2):537–581, 2009.

[28] P. E. Souganidis. Stochastic homogenization of Hamilton-Jacobi equations and some applications. *Asymptot. Anal.*, 20(1):1–11, 1999.

Ceremade, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris cedex 16 - France
E-mail address: cardaliaguet@ceremade.dauphine.fr

Ecole des Ponts and Inria, 6 -8 avenue Blaise Pascal, Cite Descartes, Champs-sur-Marne, 77455 Marne La Vallee cedex 2 - France
E-mail address: lebris@cermics.enpc.fr

Department of Mathematics, University of Chicago, Chicago, Illinois 60637, USA
E-mail address: souganidis@math.uchicago.edu